

SOVIET PHYSICS JETP

A translation of the Zhurnal Éksperimental'noi i Teoreticheskoi Fiziki

Vol. 14, No. 3 pp. 485-703

(Russ. orig. Vol. 41, No. 3, pp. 673-991, Sept., 1961)

March, 1962

EXCHANGE INTERACTION AND MAGNETO-OPTICAL EFFECTS IN FERRITE GARNETS

G. S. KRINCHIK and M. V. CHETKIN

Moscow State University

Submitted to JETP editor March 17, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 673-680 (September, 1961)

It is shown that in the infrared region of the spectrum, along with the Faraday effect due to electronic transitions, there exists a Faraday effect due to magnetic spin resonance, that is, to a precession of the magnetization vector under the action of the magnetic field of the light wave. In ferrite garnets containing a magnetoactive rare-earth sublattice this effect is also determined by exchange resonance, that is, by precession of the magnetization vector of one sublattice in the exchange field of the other. In europium ferrite garnet the Zeeman effect on the line ${}^7F_0 \rightarrow {}^7F_6$ is observed in an exchange field of 2.5×10^5 oe. In this case the Faraday effect is due to exchange and not to spin-orbit splitting.

1. EXCHANGE RESONANCE IN FERRITES AND PRECESSION OF THE SPONTANEOUS MAGNETIZATION VECTOR AT OPTICAL FREQUENCIES

MEASUREMENTS of the Faraday effect in yttrium garnet^[1] have shown that rotation of the plane of polarization is independent of λ in the region $4\mu < \lambda < 9\mu$. It has been assumed that this rotation is an exclusive property of ferromagnetic semiconductors. In the first part of this paper we shall show that the Faraday effect in the transparency region of a ferrite garnet is actually of purely magnetic origin. The frequency independent rotation of the plane of polarization of infrared light is due to the precession of the spontaneous magnetization vector of the ferromagnetic semiconductor under the influence of the magnetic vector of the light wave, i.e., it is a consequence of magnetic spin resonance. The characteristic frequencies of this resonance ω_0 occur in the region of significantly longer wavelengths; we shall therefore be interested in the theory of the Faraday effect for $\omega \gg \omega_0$.

The theoretical treatment of the motion of the ferromagnetic spin system in crossed constant and alternating magnetic fields is based on the well-known Landau-Lifshitz equation.^[2] In particular, the Faraday effect, i.e., the rotation of the plane of polarization of the electromagnetic wave in the magnetization direction of the ferromagnet, is determined by the magnitude of the off-diagonal component of the permeability tensor $M = i\mu_{xy} = -i\mu_{yx}$, an expression for which can be obtained from the solution of the Landau-Lifshitz equation. From Eq. (38) of ^[2] we obtain for $\omega \gg \omega_0$ the following expression for $\tilde{M} = M_{\omega \gg \omega_0}$:

$$\tilde{M} = (4\pi\gamma/\omega) I_s, \quad (1)$$

where I_s is the saturation magnetization of the ferromagnet and $\gamma = e/mc$.

Off-diagonal components of the permeability tensor different from zero lead to a specific Faraday rotation equal to

$$\alpha_F = \frac{V_E}{2c} \omega M$$

Substituting $M = \tilde{M}$ from Eq. (1), we have

$$\tilde{\alpha}_F = \frac{2\pi V_E}{c} \gamma I_s \text{ rad/cm}$$

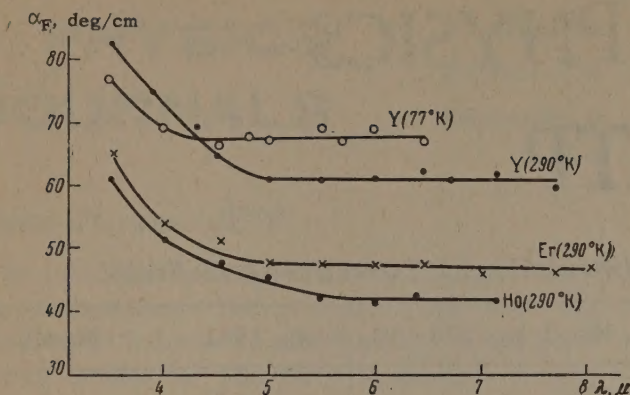


FIG. 1. Faraday effect in yttrium, erbium, and holmium garnets in infrared light at different temperatures; $H_{\text{ext}} = 2400$ oe.

The Faraday effect in the microwave region has been found in semiconducting ferrites, investigated in a number of papers, and utilized in microwave technology. Up until now, however, no attention has been paid to the circumstance that magnetic spin resonance should produce the same Faraday effect for infrared and visible light as it does for electromagnetic waves in the microwave region, because $\tilde{\alpha}_F$ is independent of ω . The magnitude of the specific rotation deduced from Eq. (2) amounts to several tens of degrees per centimeter.

The investigation of the rotation of the plane of polarization of light is of interest also because by this means one can study the phenomenon of exchange resonance. As Kaplan and Kittel have shown,^[3] exchange resonance, i.e., magnetic spin resonance in the effective field of the exchange interaction between the magnetic sublattices, can occur in uncompensated antiferromagnets. The intensity of the exchange resonance is proportional to $(\gamma_1 - \gamma_2)^2$, where $\gamma_1 = g_1 e / 2mc$ and $\gamma_2 = g_2 e / 2mc$ are the gyromagnetic ratios of the ions on the two magnetic sublattices.

The exchange resonance frequency $\omega_0^{\text{exch}} = \lambda \times (\gamma_2 I_1 - \gamma_1 I_2)$, where λ is the molecular field coefficient, and I_1 and I_2 are the spontaneous magneti-

zations of the two sublattices. For values of $H_{\text{exch}} = \lambda I = 10^6$ oe, the magnitude of $\omega_0^{\text{exch}} = 3 \times 10^{12}$ cps, i.e., the resonance frequency is in the far infrared region. Because of the technical difficulties in this region, direct observation of exchange resonance has so far not been reported. Effects associated with exchange resonance have been observed only near the compensation point, since then $I_1 \approx I_2$ and the frequency ω_0^{exch} decreases and shifts toward the more convenient microwave region.

Let us consider how the exchange resonance influences the magnitude of $\tilde{\alpha}_F$. Wangsness^[4] has obtained expressions for the tensor components of the magnetic susceptibility of a ferrimagnet with two magnetic sublattices. Using his Eq. (16) and assuming $\omega_0^{\text{res}} \ll \omega \ll \omega_0^{\text{exch}}$, we obtain expressions for $\tilde{M}$ and $\tilde{\alpha}_F$ valid in the absence of exchange resonance (ω_0^{res} is the characteristic frequency of ordinary ferromagnetic resonance):

$$\tilde{\alpha}_F^{\text{res}} = \frac{2\pi V \epsilon}{c} \gamma_{\text{eff}} I, \quad (3)$$

where $\gamma_{\text{eff}} = (I_1 - I_2) / (I_1 / \gamma_1 - I_2 / \gamma_2)$ is the ratio of the total magnetic moment of the ferrite to its mechanical moment. For frequencies $\omega \gg \omega_0^{\text{res}}$, ω_0^{exch} the applicable formula in the presence of exchange resonance is

$$\tilde{\alpha}_F^{\text{exch}} = \frac{2\pi V \epsilon}{c} (\gamma_1 I_1 - \gamma_2 I_2). \quad (4)$$

When $\gamma_1 = \gamma_2$, (3) and (4) reduce to (2). The physical interpretation of these expressions is that during ordinary ferromagnetic resonance both sublattices of the ferrimagnet, bound together by the exchange forces, precess as a single unit, whereas during the exchange type of precession each sublattice makes an independent contribution to the Faraday effect, with a sign corresponding to the orientation of the magnetic moment of the sublattice. The equations for ferrimagnets with three magnetic sublattices have a similar form. For the sake of simplicity, we shall employ Eqs.

Garnet	n	Ground level of Ion		g_1	g_2	g_{eff}	T, °K	H, oe	I_1	I_2	$\tilde{\alpha}_F^{\text{res}}$	$\tilde{\alpha}_F^{\text{exch}}$	$\tilde{\alpha}_F$
		Sub-lattice 1	Sub-lattice 2								theory	experiment	
$Y_3Fe_5O_{12}$	2.2	$^6S_{7/2}$	—	2.0	—	2.0	290	2400	135	—	63 *	—	61
							77	2400	181	—	84	—	68
							77	2800	181	—	84	—	74
$Ho_3Fe_5O_{12}$	2.2	$^6S_{7/2}$	$^5I_8$	2.0	1.25	8.5	290	2400	135	75.8	117	41	42
$Er_3Fe_5O_{12}$	2.2	$^6S_{7/2}$	$^4I_{15/2}$	2.0	1.2	3.1	290	2400	135	46.8	63	50	47

*Values of $\tilde{\alpha}$ are given in deg/cm.

(3) and (4), since $g_1 \approx g_2 \approx 2$ for the iron sublattices in ferrite garnet, and the improvements in accuracy of the values of $\tilde{\alpha}_F$ by taking into account the difference between g_1 and g_2 exceed the limits of precision of the measurements.

Measurements of the Faraday effect in yttrium, holmium, and erbium garnets were carried out by the method described earlier^[1,5] at wavelengths of from 3.5 to 8 μ . Measurements were made with an NaCl prism having a slit width of 2 mm using a vacuum thermocouple as a detector. The measurements on the yttrium garnet were made quantitatively more precise than the preliminary data,^[1] and in addition were made at liquid-nitrogen temperature.

The results of the measurements are compared with theory in Fig. 1 and the table. The theoretical values for all the garnets were calculated for a dielectric constant $\epsilon = 4.48$. Data on the index of refraction of the rare-earth garnets are lacking in the literature. We measured the reflection coefficient for light of $\lambda = 0.6 \mu$ and $\lambda = 1.0 \mu$ from natural faces of monocrystals of yttrium and erbium garnets. The value of $n = 2.2 \pm 0.2$ was obtained for both garnets. The curves of $\tilde{\alpha}_F$ for yttrium garnet were taken in an external field $H = 2400$ oe. At room temperature the sample is saturated in this field, since the demagnetization field $4\pi I_S \approx 1700$ oe. At liquid-nitrogen temperature $4\pi I_S \approx 2300$ oe; hence, although the Faraday effect also increases, it still does not attain the calculated values corresponding to magnetic saturation of the sample. By forcing the electromagnetic additional measurements were made at 2800 oe, which were in more satisfactory agreement with the calculated values for $\tilde{\alpha}_F$ (see table).

The data show that the experimental values of $\tilde{\alpha}_F$ for erbium and holmium garnets agree with theory only under the assumption that exchange resonance is present in these garnets. The differences between the experimental values for $\tilde{\alpha}_F$ and the theoretical values for $\tilde{\alpha}_F^{\text{res}}$ calculated from Eq. (3) for ordinary ferromagnetic resonance lie far outside the limits of possible experimental errors. A change in sign of $\tilde{\alpha}_F$ was also observed for holmium garnet upon cooling the sample to liquid-nitrogen temperature ($\lambda \approx 5 \mu$), which is a further indication of the presence of exchange resonance in rare-earth garnets. Measurements of $\tilde{\alpha}_F$ at different temperatures in dysprosium and terbium garnets would be of interest, since according to Eq. (3) the specific rotations $\tilde{\alpha}_F^{\text{exch}}$ and $\tilde{\alpha}_F^{\text{res}}$ should have completely different temperature dependences in these garnets.

Thus, comparison of all the experimental results with theory leads to the conclusion that in the wavelength interval studied rotation of the plane of polarization of light by a ferromagnet is caused by its dynamic magnetic susceptibility in the alternating magnetic field of the light wave. The Faraday effect measured amounts to several tens of degrees per centimeter, i.e., it is of the same order of magnitude as the Faraday effect in ferrites at microwave frequencies. A similar effect should be observable in transparent ferromagnets and in sufficiently strongly magnetized paramagnets in a wide frequency interval, 10^9 to 10^{17} cps. It should be mentioned that the detection at very high frequencies of effects associated with the diagonal tensor components of the magnetic susceptibility tensor is practically impossible, since these components decrease in proportion to the square of the frequency.

A Faraday effect due to electronic transitions under the influence of the electric vector of the light wave becomes dominant in the transition from infrared to visible light. It is possible to isolate the wavelength region where the rotation of the plane of polarization evoked by the tensor character of the permeability approximately equals the rotation brought about by the tensor character of its dielectric constant, i.e., the garnet ferrite is a bigyrotropic medium. For some wavelengths these rotations will have the same sign and the total Faraday effect will increase; for others they will be of different sign and the net effect reduces to zero.^[5]

The good quantitative agreement of the experimental and theoretical values of $\tilde{\alpha}_F$ indicates that measurements of the Faraday effect in the infrared region can be utilized for the determination of the magnitude, anisotropy, and temperature dependence of the g-factor of rare-earth ions contained in garnet ferrites. All the remaining quantities in Eq. (4) can be determined fairly easily from independent measurements.

2. THE ZEEMAN EFFECT IN THE EXCHANGE FIELD OF THE FERROMAGNET

In this section the results of an investigation of the fine structure of one of the infrared absorption lines of Eu^{3+} ion in the garnet ferrite $\text{Eu}_3\text{Fe}_5\text{O}_{12}$ will be stated. The europium ion was selected for study because its ground level ${}^7F_0$ is not split by the crystalline field and the fine structure of the line arises only from the splitting of the higher level. According to the energy-level scheme of Eu^{3+} ,^[6] a number of lines corresponding to elec-

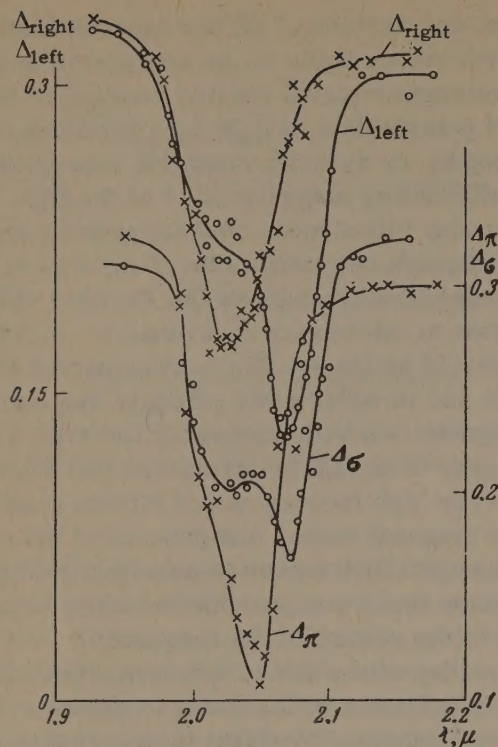


FIG. 2. Transparency of europium garnet plates $100\ \mu$ thick in a magnetic field $H = 2000$ oe (Δ is the ratio of the intensities of transmitted and incident light). Δ_{right} — the case of right-circularly polarized infrared light (electric vector $\mathbf{e}$ of the light wave rotates clockwise, looking along the light ray), magnetic field directed along the light beam; Δ_{left} — the same for left-circularly polarized light. Δ_{π} — linearly polarized light, $\mathbf{e} \parallel \mathbf{H}$, magnetic field parallel to surface of plate; Δ_{σ} — the same for $\mathbf{e} \perp \mathbf{H}$.

tronic transitions between ground term levels should occur in the infrared region. The absorption line corresponding to the transition ${}^7F_0 \rightarrow {}^7F_6$ should lie close to $\lambda = 2\ \mu$ according to this scheme. We have actually found this line; the polarization of its components turns out to be directly connected with the orientation of the magnetization vector of the garnet ferrite.

The measurements were made in an IKS-11 spectrograph with a LiF prism having a slit width of $0.2\ \text{mm}$, which corresponds to a spectral interval of $\sim 40\ \text{cm}^{-1}$. A Fresnel rhomb was used to obtain light of circular polarization. Just as in the remaining cases, the sample had the form of a polished monocrystalline plate $100\ \mu$ thick. A lead-sulfide cell cooled to liquid nitrogen temperature was used as the light detector.

Figure 2 shows the results of these measurements. With longitudinal magnetization of europium garnet in a magnetic field of 2000 oe (light directed parallel to the magnetization vector and perpendicular to the surface of the sample), one observes two absorption line components split by

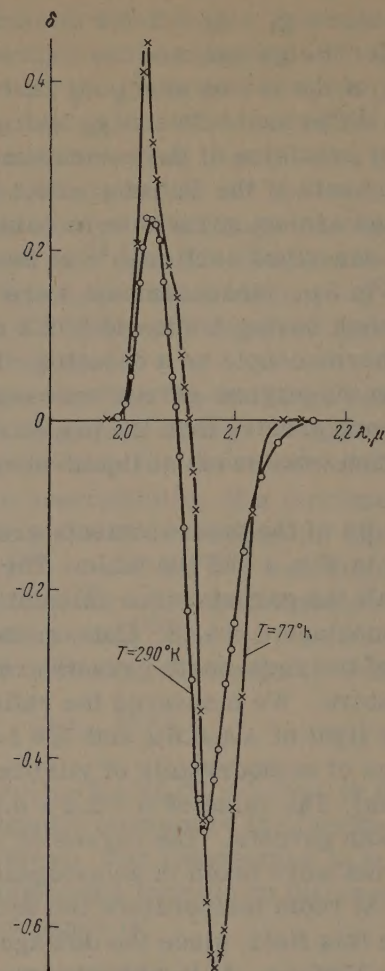


FIG. 3. Circular dichroism $\delta = (\Delta_{\text{right}} - \Delta_{\text{left}})/\Delta_{H=0}$ of europium garnet in the absorption band region.

$110\ \text{cm}^{-1}$, corresponding to the right and left circular polarizations of the infrared light.

A change in the sign of the magnetization vector of the sample is equivalent to a change in sign of the circular polarization of the light. This assertion is illustrated in Fig. 3 by the curves of the circular dichroism of europium garnet, which were obtained by reversing the magnetizing field. With transverse magnetization of the sample (light directed perpendicular to the sample surface, magnetization vector lies in the plane of the sample) and linear polarization, one observes one component (π) for light with the electric vector parallel to the magnetization of the sample, and two shifted components (σ) of different intensity for light of the other polarization.

Thus, in the resolution obtained, the observed spectrum has the characteristics of a Zeeman triplet with a splitting corresponding to a magnetic field of several hundred thousand oersteds. It is natural to compare this field with the exchange field acting on a europium ion, arising from the iron magnetic sublattices. According to Wolf and

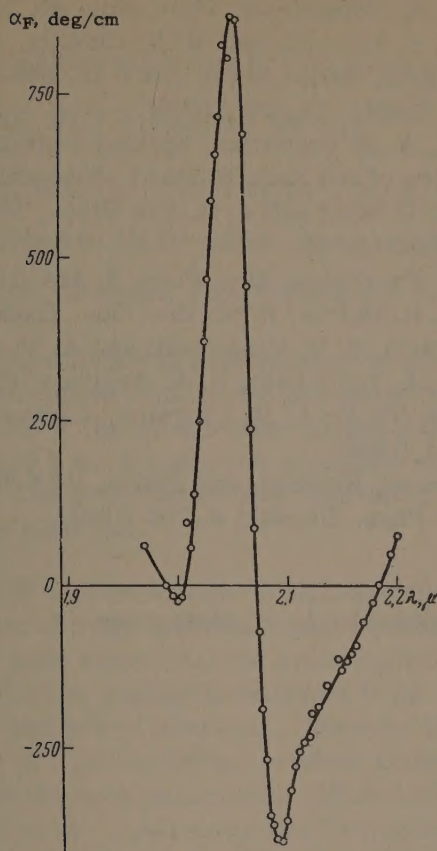


FIG. 4. Faraday effect in europium garnet in the absorption band region.

Van Vleck,^[7] this field amounts to $\sim 2.5 \times 10^5$ oe at room temperature. Substituting the values $g = 1.5$, $J = 6$, $H = 2.5 \times 10^5$ into the formula for the total Zeeman splitting of the level of a free ion with given J ($2\Delta E = 2gJ\mu_B H$), we obtain $2\Delta E \approx 210$ cm^{-1} .

Considering that the observed absorption lines reflect a neutralized picture of the Zeeman components displaced to the right and left, the experimental data agrees completely with the quoted estimate. One should not expect too much from such an estimate, since in the development of the theory of the exchange Zeeman effect in ferromagnetic crystals it is necessary to take into account the Stark splitting of the levels, the effects that eliminate the forbiddenness of electronic transitions between f-shell levels, and the selection rules with respect to the crystalline quantum number. A comparison of the curves of circular dichroism at room temperatures (Fig. 3) shows that the splitting is 30% greater at 77° K than it is at 290° K. This fact is consistent with the assumption that the splitting arises from the exchange field, since, according to Pauthenet's data,^[8] the magnetization of the iron sublattices in garnet ferrites increases by just this amount in going from room to nitrogen temperature.

On the basis of these results it is possible to reveal one of the physical mechanisms responsible for the magneto-optical properties of ferromagnets. In fact, Fig. 4 shows the results of measurements of the Faraday effect in europium garnet in the vicinity of the absorption line under study. The curve has the typical resonance form, as in other garnet ferrites near the absorption bands.^[5] However, in the present case it is possible to point directly to the physical cause of the rotation of the plane of polarization, namely the exchange splitting of an excited energy level of a rare-earth ion. The exchange splitting causes the circular dichroism and double circular refraction, i.e., the difference in the absorption coefficients and refractive indices for right- and left-circularly polarized light, and consequently, also the Faraday effect. In the development of the microscopic theory of magneto-optical effects in ferromagnets it has been clear from the very beginning that the anomalously large magnitude of the magneto-optical effects in ferromagnets was connected with the action of a huge "internal field" of the order of 10^5 to 10^7 oe. However, in all quantitative theories of magneto-optical effects in ferromagnetic metals and semiconductors^[9] it has been assumed that this field originates from a spin-orbit interaction. It appears to us that the fundamental role is played not by spin-orbit, but by exchange splitting of the energy levels, at least in the magneto-optics of rare-earth garnet ferrites.* The spin-orbit splitting in this case determines only the natural frequency of the resonance curve of the Faraday effect, since the splitting of the ground term in rare-earth ions is determined by the spin-orbit interaction. It is possible that the suggested mechanism of exchange splitting explains the magneto-optical properties of ferromagnets in other cases as well.

It is necessary to pay attention to one other peculiarity of the phenomenon. In essence, we have the possibility of controlling the appearance of polarized absorption lines in the spectrum of a solid by means of a weak external magnetic field. Because the orbital magnetic moment of the f-shell of a rare-earth ion is bound to the spin, a constant external magnetic field, by changing the orientation of the spontaneous mag-

*We have also detected circular dichroism in the vicinity of the absorption bands in erbium and holmium garnets. The separations between the maxima of the effects of different sign, which depend on the magnitude of the splitting, have the same order of magnitude as in the case of europium garnet. The absolute values of the effect were smaller (~ 3 to 10%) than in the europium garnet, apparently owing to the lack of resolution of the separate components of the absorption bands.

netization vector, alters the "population" of some of the energy levels that play an important role in electrical dipole transitions. Hence, one cannot exclude the possibility that the exchange Zeeman effect will find application in the construction of a kind of molecular amplifier in the optical region or of quantum light indicators.^[10] Of importance here is the circumstance that the exchange splitting of the levels is very great, permitting the attainment of energy level "populations" close to 100% at relatively high temperatures.

It is a pleasure to thank G. A. Smolenskii for preparing the europium garnet single crystals.

¹G. S. Krinchik and M. V. Chetkin, JETP **38**, 1643 (1960), Soviet Physics JETP **11**, 1184 (1960).

²L. D. Landau and E. M. Lifshitz, Sow. Phys. **8**, 153 (1935).

³J. Kaplan and C. Kittel, J. Chem. Phys. **21**, 760 (1953).

⁴R. K. Wangsness, Phys. Rev. **95**, 339 (1954).

⁵G. S. Krinchik and M. V. Chetkin, JETP **40**, 729 (1961), Soviet Phys. JETP **13**, 509 (1961); Opt. i Spekr. **11**, 274 (1961).

⁶M. A. El'yashevich, Spektry redkikh zemel' (Spectra of the Rare Earths), Gostekhizdat, 1953.

⁷W. P. Wolf and J. H. Van Vleck, Phys. Rev. **118**, 1490 (1960).

⁸R. Pauthenet, Ann. Phys. **3**, 424 (1958).

⁹H. R. Hulme, Proc. Roy. Soc. (London) **A135**, 237 (1932); S. V. Vonsovskii and A. V. Sokolov, JETP **19**, 703 (1949); P. N. Argyres, Phys. Rev. **97**, 334 (1955); A. M. Clogston, J. phys. radium **20**, 151 (1959).

¹⁰Basov, Krokhin, and Popov, UFN **72**, 161 (1960) Soviet Phys.-Uspekhi **3**, 702 (1961).

Translated by L. M. Matarrese
121

MEASUREMENTS OF THE LONGITUDINAL POLARIZATION OF ELECTRONS EMITTED IN THE BETA DECAY OF Au^{198}

R. O. AVAKYAN, G. L. BAYATYAN, M. E. VISHNEVSKII, and E. V. PUSHKIN

Submitted to JETP editor March 21, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 681-683 (September, 1961)

The dependence of polarization on β -electron energy is investigated for Au^{198} . Polarization measurement results are presented (in units of v/c) for Au^{198} β electrons relative to Sm^{153} . For Au^{198} the polarization differs significantly from unity at β -electron energies of 60, 90, and 120 kev. A spherical capacitor was used to change longitudinal into transverse polarization. The polarization was calculated from the asymmetry of β -electron scattering on a heavy element.

NUMEROUS papers devoted to the experimental investigation of longitudinal electron polarization in β decay have shown that the polarization is either equal to or negligibly different from v/c^* for a large number of elements. However there exist cases of first-forbidden β -decay transitions in which the electron polarization differs appreciably from unity. Apparently this deviation is strongly correlated with the deviation of the β spectrum from the Fermi shape.^[1] The classic example of this type is the β decay of RaE , whose electron polarization is considerably different from unity^[2] and whose β -spectrum shape differs from the Fermi shape. Both of these anomalies are explained by the particular structure of this nucleus.

Another example is Au^{198} (the $2^+ \rightarrow 2^-$ transition), whose electron polarization equals unity at energies greater than 200 kev but deviates from unity at low energies (100 – 150 kev).^[3-6]

We measured the longitudinal polarization of Au^{198} β electrons in the 60 – 120-kev energy range using the method of Mott scattering. The longitudinal polarization was converted into transverse polarization by deflecting the electrons in the electric field of a spherical capacitor.

DESCRIPTION OF THE EXPERIMENTAL ARRANGEMENT

A schematic diagram of the apparatus is presented in Fig. 1. Electrons from the β -active source are collimated and enter the electrostatic spectrometer (a spherical capacitor which deflects the electrons by 108° and has an energy resolution of $\Delta E/E = \pm 0.07$).

*The polarization is measured everywhere below in units of v/c .

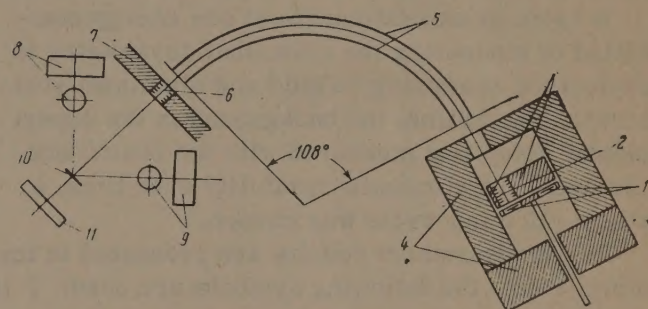


FIG. 1. Schematic diagram of the apparatus: 1 – β sources; 2, 6 – collimators; 3, 4, 7 – lead shielding; 5 – spectrometer; 8, 9 – Geiger counter telescopes; 10 – scatterer; 11 – scintillation counter.

The electron beam from the spectrometer strikes the scatterer. Electrons scattered through 112° – 141° are recorded by two counter telescopes, each consisting of two Geiger counters connected in coincidence.

In order to carry out relative measurements, two sources are inserted simultaneously into the apparatus. The construction of the device permits one to change scatterers and sources without breaking the vacuum and to apply high voltage to the source in order to vary the β -electron energy.

The sources were prepared in the following manner: a thin film of gold was sputtered onto an aluminum foil in vacuum; the samarium was deposited on an aluminum foil by the precipitation method. A silicon organic resin was used as a binding agent. The sources thus prepared were activated in a reactor.

MEASUREMENTS

The longitudinal polarization of β electrons was measured for Au^{198} at 60, 90, and 120 kev, and for Sm^{153} at 120 kev.

T, kev	Element	Δ	P_{rel}	P_{abs}
60	Au ¹⁹⁸	0.134 ± 0.016	0.60 ± 0.07	0.56 ± 0.06
90	Au ¹⁹⁸	0.181 ± 0.018	0.73 ± 0.08	0.68 ± 0.08
120	Au ¹⁹⁸	$0.226 \pm 0.017^*$	0.80 ± 0.07	0.74 ± 0.07
120	Sm ¹⁵³	0.257 ± 0.016	1	0.93 ± 0.06

*The value presented is corrected for the presence of conversion electrons.

During the work a constant voltage was applied to the spherical capacitor which separated out electrons of 90 ± 6 kev energy. Thus, in all the experiments electrons of the same energy struck the scatterer, and all corrections to azimuthal asymmetry in the scattering were the same irrespective of the β -spectrum region chosen. In order to measure the polarization of 60- or 120-kev β electrons, an accelerating or braking electric field of 30 kev was applied between source and condenser.

A cycle of measurements at one energy consisted of measuring the azimuthal asymmetry in β -electron scattering on gold and aluminum scatterers. In addition, the background in the experimental setup was measured with the scatterers removed. To eliminate instability over time, a short (15 min) cycle was chosen.

The measurement results are presented in the table, where the following symbols are used: T is the kinetic energy of the electrons, Δ is the azimuthal asymmetry in electron scattering from 0.13 mg/cm² of gold, and P is the longitudinal polarization of the electrons.

In calculating the β -electron polarization of Au¹⁹⁸ relative to Sm¹⁵³, only the corrections for depolarization in the sources and for internal conversion electrons are important. All other corrections associated with the relative nature of the measurements are eliminated.

The absolute value of the polarization is $P = \Delta/S$, where Δ is the azimuthal asymmetry of the scattering, measured experimentally, and S is the scattering asymmetry of a completely polarized beam.

To determine S we made use of Sherman's results,^[7] introducing corrections for the screening and Z of the nucleus as well as for multiple scattering in the scatterer.^[8] Depolarization in the source varied between 1 and 10% in the various experiments. Apparatus asymmetry, eliminated by substituting an aluminum scatterer for the gold one, did not exceed 6%. A spectrum of the source Au¹⁹⁸ + Au¹⁹⁹ in the 30–160-kev energy region was taken on our apparatus (Fig. 2) in order to investigate the presence of conversion electrons in the given energy intervals. Here the correction to the

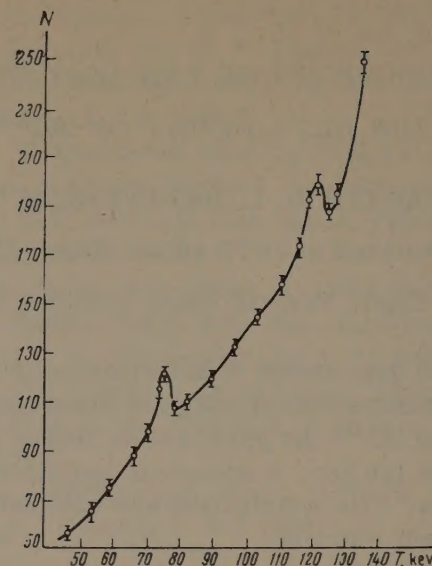


FIG. 2. Spectrum of conversion electrons and β electrons from Au¹⁹⁸ + Au¹⁹⁹.

polarization was 0.05 for an average energy of 120 kev. At 60 and 90 kev no noticeable contribution from internal conversion electrons was observed. The measurements showed that the contribution from electrons multiply scattered on the condenser plates was (1–2)%, and that from electrons back-scattered by the walls of the source chamber less than 1%.

The longitudinal polarization of Au¹⁹⁸ β electrons in the 60–120-kev energy range as measured by us differs appreciably from unity. This difference increases with decrease in energy. The deviation obtained in the polarization value is not unexpected,^[1] since the β spectrum of Au¹⁹⁸ apparently does not have a Fermi shape at energies lower than 250 kev.^[9]

The authors thank A. I. Alikhanov, G. P. Eliseev, and V. A. Lyubimov for discussing the results of the work, and also V. M. Berezin for preparing and adjusting the separate units of the apparatus.

¹B. V. Geshkenbein and A. P. Rudik, JETP 38, 1894 (1960), Soviet Phys. JETP 11, 1361 (1960).

²Alikhanov, Eliseev, and Lyubimov, JETP 35, 1061 (1958), Soviet Phys. JETP 8, 740 (1959).

³Alikhanov, Eliseev, and Lyubimov, JETP 39, 587 (1960), Soviet Phys. JETP 12, 414 (1961).

⁴Vishnevskii, Grigor'ev, Ergakov, Nikitin, Pushkin, and Trebukhovskii, Yadernye reaktsii pri malykh i srednikh énergiyakh (Nuclear Reactions at Low and Medium Energies), AN SSSR, 1958, p. 336.

⁵Turner, Gard, and Cavanagh, Bull. Amer. Phys. Soc. 4, 77 (1959).

⁶ P. E. Spivak and L. A. Mikaélyan, JETP 39, 574 (1960), Soviet Phys. JETP 12, 404 (1961).

⁷ N. Sherman, Phys. Rev. 103, 1601 (1956).

⁸ Alikhanov, Eliseev, and Lyubimov, JETP 34, 1045 (1958), Soviet Phys. JETP 7, 723 (1958).

⁹ R. M. Steffen, Proc. of the Rehovoth Confer-

ence on Nuclear Structure, 1957, Amsterdam, 1958, p. 426.

Translated by Mrs. J. D. Ullman
122

SPIN DEPENDENCE OF WEAK INTERACTION IN $\mu^- + p \rightarrow n + \nu$

L. B. EGOROV, G. V. ZHURAVLEV, A. E. IGNATENKO, A. V. KUPTSOV, LI HSÜAN-MING, and M. G. PETRASHKU

Joint Institute for Nuclear Research

Submitted to JETP editor March 31, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 684-691 (September, 1961)

The asymmetry coefficient $\bar{a}_0$ of μ -e decay electrons averaged over the two hfs states has been measured with scintillation counters for mesonic atoms of silver and of red and black phosphorus. The meson lifetime τ in these two modifications of phosphorus has also been measured. The relative values of a_0 for red (nonconductive) and black (conductive) phosphorus indicate directly that meson spin relaxation time in mesonic atoms is decreased by the presence of conduction electrons. The values of $\bar{a}_0$ and τ in phosphorus are used to determine the level populations for $F = 1$ and 0 , and the probabilities λ_1 and λ_0 for meson capture by the nucleus from these states. The results indicate directly that $\lambda_1 \neq \lambda_0$ (the weak interaction is spin dependent) and $\lambda_0 > \lambda_1$ (capture occurs more rapidly from the $F = 0$ state than from the $F = 1$ state). The lower limit obtained for $\Delta\lambda/\lambda = (\lambda_1 - \lambda_0)/(\frac{3}{4}\lambda_1 + \frac{1}{4}\lambda_0)$ is direct evidence for the $A - xV$ type of interaction.

1. INTRODUCTION

It has been noted in [1] that the measurement of muon capture from the two hfs states can furnish evidence regarding the spin dependence of weak interaction in the process $\mu^- + p \rightarrow n + \nu$. If the capture probabilities for the two states are different, the time dependence of the number of electrons from μ -e decay in isolated mu-mesonic atoms will not be a simple exponential function and the logarithm of the decay curve should exhibit positive curvature. This theory can be checked experimentally only by mu-mesonic atom production in matter. However, mu-mesonic transitions between the hyperfine levels, due to the presence of the medium, can complicate the picture. Telegdi [2] has shown that the existence of transitions and the spin dependence of the interaction can be used in experimental investigations of the type of interaction involved. When the transition probability R is known, the sign and magnitude of the curvature K of the logarithmic decay curve determine the interaction type uniquely. However, the determination of R is very complicated [3-5] and has not yet been accomplished experimentally. [6] It is therefore impossible to arrive at final conclusions regarding the interaction type solely on the basis of the experimental value of K . [6]

Überall [5] and Primakoff [7] have shown that the interaction type can also be determined uniquely by measuring n_1 and n_0 , the populations of the states

with total angular momenta $F = 1$ and 0 , respectively, and $\Delta\lambda$, the difference between muon capture probabilities from these states. The nucleus of phosphorus, having spin $I = \frac{1}{2}$, is a convenient object for such investigations. Among the allotropic modifications red (nonconductive) phosphorus and black (conductive) phosphorus are of greatest interest, since conduction electrons perform a "catalytic" function by reducing the relaxation time of nuclear spin. It is expected [2,5,7] that because of conversion associated with conduction electrons the difference between n_1 and n_0 for these modifications at the time of muon capture will be sufficient to permit the measurement of these quantities and of $\Delta\lambda$.

In the present work we have determined n_1 , n_0 , and $\Delta\lambda$ by measuring the asymmetry of the angular distribution of electrons from μ -e decay and the μ^- -meson lifetime, in red and black phosphorus.

2. EXPERIMENT

a) Asymmetry of the μ -e decay electron distribution. When the method of [8] is used for the asymmetry measurement the experimental precession curve for nuclei with $I \neq 0$ will represent the superposition of the precession curves of muons decaying from both hfs states. Therefore $\bar{a}_0$, the asymmetry coefficient in the angular distribution $1 + a_0 \cos \theta$ integrated over electron energies, will be the average for the two states. $\bar{a}_0$

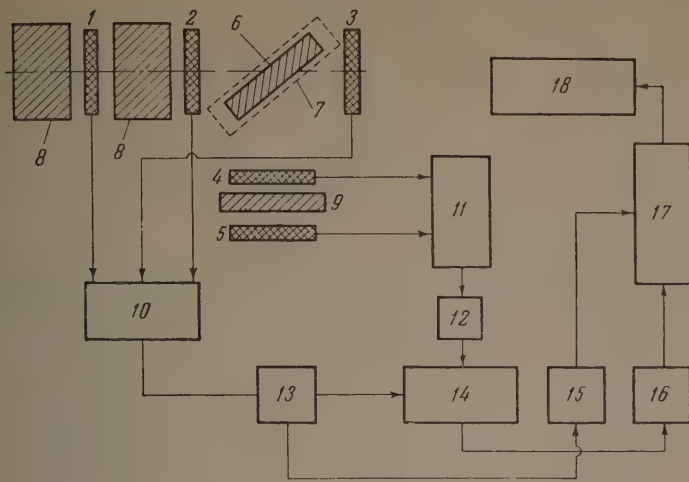


FIG. 1. Block diagram of apparatus. 1–5 – scintillation counters; 6 – target; 7 – magnetizing coil; 8 – copper absorbers; 9 – paraffin absorber; 10 – anticoincidence circuit; 11 – coincidence circuit; 12 – delay line; 13 – gate trigger; 14 – gate; 15, 16 – pulse shapers; 17 – converter; 18 – amplitude analyzer.

will be time dependent because of transitions.^[5,6] Relative measurements^[9,10] averaged over the entire spectrum of a_0 for $I = 0$ show that a_0 is independent of time and the atomic number Z .

If a_0 is independent of Z , a comparison of $\bar{a}_0$ for silver and phosphorus can indicate the populations of the levels $F = 1$ and 0 at the time of muon capture by phosphorus nuclei. In mesonic silver atoms having $I = 1/2$ and nuclear magnetic moment $\mu_N < 0$, transitions will proceed from $F = 0$ to $F = 1$. Since muons are depolarized in the state $F = 0$, while the state $F = 1$ “remembers” the spin direction, transitions do not affect $\bar{a}_0$, which will be constant and equal to $1/2 a_0$.^[11] In mesonic phosphorus atoms having $I = 1/2$ and $\mu_N > 0$, the state $F = 0$ lies lower and conversions will reduce $\bar{a}_0$ with time. For phosphorus at time $t = 0$ we have $a_0 = \bar{a}_{Ag} = 1/2$. Then the effective mean level populations $\bar{n}_1$ and $\bar{n}_0$ for the time interval from 0 to t can be determined (taking $\bar{n}_1 + \bar{n}_0 = 1$ into account) from the relation

$$\bar{a}_P / \bar{a}_{Ag} = \bar{a}_P / \bar{a}_0 = (\bar{n}_1 a_1 + \bar{n}_0 a_0) / \left(\frac{3}{4} a_1 + \frac{1}{4} a_0 \right), \quad (1)$$

where $\bar{a}_P$ and $\bar{a}_{Ag}$ are the values of $\bar{a}_0$ for phosphorus and silver measured during the time from 0 to t , and a_1 and a_0 are the asymmetry coefficients in the $F = 1$ and 0 states depending only on the nuclear spin.^[11]

Sulfur and cadmium, which have $I = 0$ and Z close to phosphorus and silver, respectively, can be used as controls to determine whether a_0 is actually independent of Z . Mann and Rose^[12] have predicted theoretically that a_0 should be independent of Z for $Z \geq 15$, when the absolute

Element	I	ξ
Cadmium	0	1.10 ± 0.01
Silver	$1/2$	1.05 ± 0.01
Sulfur	0	1.10 ± 0.01
Red phosphorus	$1/2$	1.04 ± 0.01
Black phosphorus	$1/2$	1.00 ± 0.01

quantum yields of the K and L series in mesonic atoms are constant^[13] and Auger conversions play only an insignificant role. a_0 and $\bar{a}_0$ can be determined by measuring the numbers of electrons $N_{\max}$ and $N_{\min}$ for two values of the magnetic field $\pm H$ within which the target is located^[14] corresponding to

$$t_1 \pm \Delta t = -T/2 - \pi mc / eH,$$

where t_1 is the delay time, Δt is the gate width, and T is the spin precession period of a “free” meson (for sulfur and cadmium) or of a mesonic nucleus (for phosphorus and silver).

The experimental conditions and apparatus were the same as those used in earlier work.^[9,14] Figure 1 is a block diagram of the apparatus. Negative muons stopping in the target 6 were registered by the 1 + 2 – 3 anticoincidence circuit 10. Pulses from 10, delayed $0.1 \mu\text{sec}$, opened the gate 12 for the time $\Delta t = 1.2 \mu\text{sec}$. Pulses due to μ -e decay electrons from the 4 + 5 coincidence circuit 11 were transmitted and were registered by a scaler.

We used $15 \times 15 \text{ cm}$ targets that were 8 g/cm^2 thick. In the experiments with sulfur and phosphorus a 7 g/cm^2 paraffin absorber 9 was placed between counters 4 and 5; for silver and cadmium this was replaced with an aluminum absorber of the same thickness. The registration efficiency for γ rays under 10 Mev emitted from the target as a result of meson absorption was reduced to under 10^{-3} by the aluminum absorber. We used targets and a paraffin absorber of considerable thickness to measure asymmetry at the end of the decay-electron spectrum, where the asymmetry coefficient is larger than a_0 .

The values of $\xi = N_{\max} / N_{\min}$ are given in the third column of the table, after correction for the delay time t_1 , the gate width Δt , the muon lifetime τ , and the solid angle of the electron detector. In correcting for meson decay and capture we used values of τ for red and black phosphorus obtained in the present work (see below), while for sulfur, cadmium, and silver we used the values given by Sens.^[15]

b) Measurement of the μ^- -meson lifetime in phosphorus. When the lifetime is measured conventionally the muon decay curve observed for

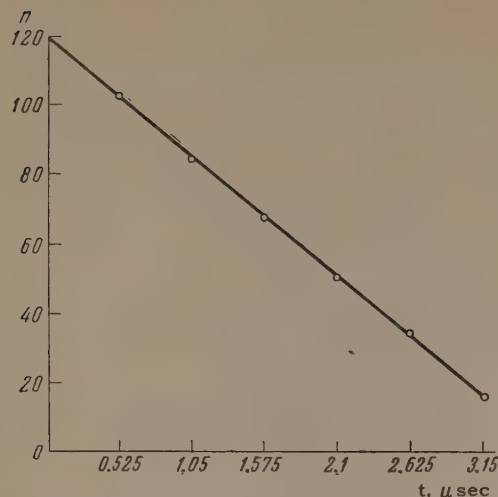


FIG. 2. Analyzer channel number n vs the delay time t between the first and second pulses in the converter.

phosphorus will represent the superposition of the muon decay curves from the two hfs states. Then the measured sum $S = \sum t_i n_i / \sum n_i$ (where n_i is the number of electrons at time t_i) will be the average over the states $F = 1$ and 0 , since it is easily shown that $S = \bar{n}_1 S_1 + \bar{n}_0 S_0$ (see Appendix I). If the level populations $\bar{n}_1$ and $\bar{n}_0$ are known for red and black phosphorus, the measurement of S will obviously determine S_1 and S_0 uniquely and also, consequently, τ_1 and τ_2 , the muon lifetimes in the states $F = 1$ and 0 .

Figure 1 is a block diagram of our apparatus. The magnetizing coil 7 compensated the stray magnetic field of the synchrocyclotron. The electronic circuitry functioned as follows. Pulses from the $1 + 2 - 3$ anticoincidence scheme 10 actuated the trigger 13, which formed positive rectangular pulses of stable $5 \mu\text{sec}$ duration. Pulses from the $4 + 5$ coincidence circuit 11 were fed through the $5 \mu\text{sec}$ gate 14; these pulses were delayed $0.2 \mu\text{sec}$ in 12 before triggering the pulse shaper 16. The delay 12 was introduced in order to register "zero" time between the pulses on the screen of the analyzer 18. Pulses designed to trigger the converter 17 reached its first input from 16 and its second input from the pulse shaper 15, which was triggered by the trailing edge of the pulse from the trigger 13. At the converter output pulses with amplitudes proportional to the time interval between the two pulses to the respective inputs were analyzed with a type AMA-3S 128-channel amplitude analyzer 18.^[16]

Linearity of the apparatus was checked with a set of RKZ-401 delay cables producing a $0.52 \mu\text{sec}$ delay. Delay equality was checked within 0.5% by the resonance method. Figure 2 shows the analyzer channel number n vs the delay t between

the two pulses at the converter. Nonlinearity is seen to be under 1% . The calibration was stable to within 1% during 15 hours of operation. The "zero" analyzer channel was determined by placing scintillation counters 1–5 along the meson beam axis, with the anticoincidence channels switched off.

The experiments with red and black phosphorus were performed under identical conditions. With black phosphorus during the time interval from 0 to $2.4 \mu\text{sec}$ (corresponding to 80 analyzer channels) 19272 electrons were registered, compared with 12088 in the case of red phosphorus. The time interval from 2.4 to $3.3 \mu\text{sec}$ was used to determine the background. At $t = 0$ the electron-to-background ratio was 23. There were 600 gate openings per second, i.e., 6 openings for each accelerator pulse.

For the calculation of S , pulses from four channels were added (for the time interval $0.127 \mu\text{sec}$). The results were $S_b = 0.540 \pm 0.007 \mu\text{sec}$ for black phosphorus and $S_r = 0.590 \pm 0.012 \mu\text{sec}$ for red phosphorus.

3. DISCUSSION OF RESULTS

The table gives equal values of ξ for sulfur and cadmium within statistical error limits. The values of a_0 obtained from ξ agree within error limits with earlier results.^[9] Equal values were also obtained for a_0 from our results for ξ for silver and red phosphorus and on the basis of a large number of points on the precession curve for red phosphorus in ^[17]. In the experiments with black phosphorus the electron counting rate was not observed to depend on the magnet coil current. In the cases of silver and red phosphorus the fact that maximum electron asymmetry is observed at a spin precession frequency of mesonic nuclei that is one-half the spin precession frequency of a free muon again ^[17] indicates directly that negative muons have spin $\frac{1}{2}$.

The equality of the measured values of a_0 for sulfur and cadmium is in agreement with the theory.^[12] Therefore ξ for silver and phosphorus can be used to determine the effective mean populations $\bar{n}_1$ and $\bar{n}_0$ of the $F = 1$ and 2 levels in the two modifications of phosphorus, using Eq. (1),

$$\bar{a}_p / \bar{a}_{Ag} = (\bar{n}_1 a_1 + \bar{n}_0 a_0) / \left(\frac{3}{4} a_1 + \frac{1}{4} a_0 \right),$$

where a_1 and a_0 equal $(2I+3)/3(2I+1)$ and $(2I-1)/3(2I+1)$, respectively.^[11] For asymmetry measurements we used the time interval from 0 to $1.3 \mu\text{sec}$ (two lifetimes τ for red phosphorus^[15]). In this time interval we had $\bar{n}_1 \sim 0$,

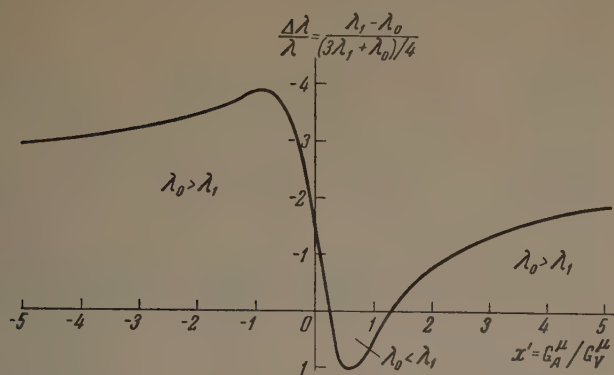


FIG. 3. $\Delta\lambda/\lambda = (\lambda_1 - \lambda_0)/((3\lambda_1 + \lambda_0)/4)$ vs $x' = G_A^\mu/G_V^\mu$ for hydrogen.

$\bar{n}_0 \sim 1$ for black phosphorus and $\bar{n}_0 \sim 2/5$, $\bar{n}_1 \sim 3/5$ for red phosphorus. These values of $\bar{n}_1$ and $\bar{n}_0$ indicate considerably different transition probabilities of muons between the hyperfine levels of mesonic atoms in these two modifications. This in turn is a direct indication that the presence of conduction electrons reduces the meson spin relaxation time in mesonic atoms. [2,7,5]

Let us now consider the measurements of S for black and red phosphorus, performed in the time interval from 0 to 2.4 μsec , which corresponds to 4τ for red phosphorus. In virtue of the fact that approximately equal numbers of μ -e decay electrons appeared in the intervals 2τ and 4τ , S can be expressed as follows for the two modifications:

$$S_r = \bar{n}_{1r}S_1 + \bar{n}_{0r}S_0, \quad S_b = \bar{n}_{1b}S_1 + \bar{n}_{0b}S_0. \quad (2)$$

Deriving S_1 and S_0 from these relations and using the tables of coefficients given by Peierls, [18] we obtain τ_1 and τ_0 , the meson lifetimes in the states $F = 1$ and 0 and, consequently, λ_1 and λ_0 , the meson capture probabilities from these states.

In the present work S_b was found to be smaller than S_r . The difference exceeds three standard deviations and is therefore significant. This fact, together with the values of $\bar{n}_1$ and $\bar{n}_0$ for red and black phosphorus, indicates directly by means of (2) that $\lambda_1 \neq \lambda_0$ (the weak interaction is spin dependent) and $\lambda_0 > \lambda_1$ (capture from the state $F = 0$ proceeds more rapidly than from the state $F = 1$).

The measurements of S_b and S_r provide a basis for definite conclusions regarding the type of weak interaction in the process $\mu^- + p \rightarrow n + \nu$, if we obtain the lower limit of

$$\Delta\lambda/\lambda = (\lambda_1 - \lambda_0)/(\frac{3}{4}\lambda_1 + \frac{1}{4}\lambda_0). \quad (3)$$

This limit is obtained from (2) with $\bar{n}_1 = \frac{3}{4}$, $\bar{n}_0 = \frac{1}{4}$ for red phosphorus and $\bar{n}_1 = 0$, $\bar{n}_0 = 1$ for black phosphorus (Appendix II). In this case we obtained

$$\begin{aligned} S_1 &= 0.61 \pm 0.016 \mu\text{sec}, & S_0 &= 0.54 \pm 0.007 \mu\text{sec}, \\ \text{upper limit } \Lambda_1 &= \lambda_1 + \lambda_{\text{decay}} = 1.42 \pm 0.035 \mu\text{sec}^{-1}, \\ \text{lower limit } \Lambda_0 &= \lambda_0 + \lambda_{\text{decay}} = 1.72 \pm 0.022 \mu\text{sec}^{-1}, \\ \text{lower limit } (\Delta\lambda/\lambda)_{\text{min}} &= -0.29 \pm 0.04; \end{aligned}$$

the meson decay probability $\lambda_{\text{decay}} = 1/\tau$ was here taken to be $4.505 \times 10^5 \text{ sec}^{-1}$.

Figure 3 shows the dependence of $\Delta\lambda/\lambda$ on $x' = G_A^\mu/G_V^\mu$ for hydrogen, calculated for the interaction type $A + xV + P$ (taking the effect of weak magnetism into account) using Primakoff's formulas. [7] For meson capture by phosphorus $\Delta\lambda/\lambda$ was calculated similarly with $x' = -1.21$ on Schmidt's model, and also on the Mayer-Jensen model, as given by Überall. [5] The results for $\Delta\lambda/\lambda$ were -0.25 and -0.45 , respectively. A comparison of these values with Fig. 3 indicates that $\Delta\lambda/\lambda$ for hydrogen is 15 or 9 times greater, respectively, than for phosphorus. $(\Delta\lambda/\lambda)_{\text{min}}$ in Fig. 3, allowing for twice the statistical error, indicates that all values $x' > 0$ (on the Schmidt model) and $x' < 0$ in the interval $0 < x' < +5$ (on the Mayer-Jensen model) are excluded, since for $x' \rightarrow +\infty$ we have $\Delta\lambda/\lambda \rightarrow -2.5$.

Our results therefore furnish direct evidence for the interaction type $A - xV$ if we take $|x'| = 1.25$ for β decay. $\Delta\lambda/\lambda$ for phosphorus is found on the basis of S and the effective mean values of $\bar{n}_1$ and $\bar{n}_0$. A comparison with theory [3,5,19] then makes it possible to obtain information regarding the probability of meson absorption by protons in different nuclear shells. We were unable to accomplish this comparison since the error in $\Delta\lambda/\lambda$ was very large as a result of the fact that much uncertainty remained regarding the values of $\bar{n}_1$ and $\bar{n}_0$ obtained by our present method.

In conclusion the authors wish to thank D. Chul'tém for assistance.

APPENDIX I

We shall prove that $S = \bar{n}_1S_1 + \bar{n}_0S_0$.

Let the sums $S_1 = \sum n_{1i} t_i / N_1$ and $S_0 = \sum n_{0i} t_i / N_0$ calculated from 0 to t correspond to the exponential functions e^{-t/τ_1} and e^{-t/τ_0} . We introduce the notation $N_1 + N_0 = N$, $N_1/N = a$, and $N_0/N = 1 - a$. Adding the sums S_1 and S_0 , we obtain

$$S = aS_1 + (1 - a)S_0 = \sum (n_{1i} + n_{0i}) t_i / N,$$

where $n_i = n_{1i} + n_{0i}$ is the total number of decays at time t_i . If τ_1 and τ_0 do not differ greatly, a and $1 - a$ will obviously equal the level populations $\bar{n}_1$ and $\bar{n}_0$, respectively.

APPENDIX II

S for the two modifications of phosphorus is expressed by

$$a_1 S_1 + (1 - a_1) S_0 = S_I, \quad a_2 S_1 + (1 - a_2) S_0 = S_{II}.$$

We now have the difference

$$S_1 - S_0 = (S_I - S_{II}) / (a_1 - a_2).$$

The derivatives of the difference $S_1 - S_0$ with respect to a_1 and a_2 are

$$\frac{\partial (S_1 - S_0)}{\partial a_1} = - \frac{S_I - S_{II}}{(a_1 - a_2)^2}, \quad \frac{\partial (S_1 - S_0)}{\partial a_2} = \frac{S_I - S_{II}}{(a_1 - a_2)^2}.$$

These expressions show that $S_1 - S_0$ decreases as a_1 increases and as a_2 decreases. The minimum of $S_1 - S_0$ will obviously be found at $a_1 \sim \bar{n}_1 = \frac{3}{4}$ and $a_2 \sim \bar{n}_2 = 0$. Taking $|S_1 - S_0| \sim |\Lambda_1 - \Lambda_0|$, we also obtain the minimum of $|\Lambda_1 - \Lambda_0|$ for $\bar{n}_1 = \frac{3}{4}$ and $\bar{n}_2 = 0$.

¹ Bernstein, Lee, Yang, and Primakoff, Phys. Rev. **111**, 313 (1958).

² V. L. Telegdi, Phys. Rev. Letters **3**, 59 (1959).

³ E. Lubkin, Phys. Rev. **119**, 815 (1960).

⁴ A. E. Ignatenko, JETP **38**, 1515 (1960), Soviet Phys. JETP **11**, 1093 (1960).

⁵ H. Überall, Phys. Rev. **121**, 1219 (1961).

⁶ V. L. Telegdi, Proc. of the Tenth Annual Rochester Conference on High-Energy Physics (Interscience Publ., Inc., New York, 1960), p. 713.

⁷ H. Primakoff, Revs. Modern Phys. **31**, 802 (1959).

⁸ Garwin, Lederman, and Weinrich, Phys. Rev. **105**, 1415 (1957).

⁹ Ignatenko, Egorov, Khalupa, and Chultém, JETP **35**, 894 and 1131 (1958), Soviet Phys. JETP **8**, 621 and 792 (1959).

¹⁰ A. E. Ignatenko, Nuclear Phys. **23**, 75 (1961).

¹¹ H. Überall, Phys. Rev. **114**, 1640 (1959).

¹² R. A. Mann and M. E. Rose, Phys. Rev. **121**, 293 (1961).

¹³ M. B. Stearns and M. Stearns, Phys. Rev. **105**, 1573 (1957).

¹⁴ Egorov, Zhuravlev, Ignatenko, Li, Petrashku, and Chultém, JETP **40**, 391 (1961), Soviet Phys. JETP **14**, 282 (1961); Nuclear Phys. **23**, 62 (1961).

¹⁵ J. C. Sens, Phys. Rev. **113**, 679 (1959).

¹⁶ Vyazemskii, Kazarinov, and Trifonov, Izvestiya LÉTI (Bulletin of Leningrad Electrotechnical Inst.) No. 30, 1959.

¹⁷ Egorov, Ignatenko, and Chultém, JETP **37**, 1517 (1959), Soviet Phys. JETP **10**, 1077 (1960).

¹⁸ R. Peierls, Proc. Royal Soc. (London) **A149**, 467 (1935).

¹⁹ L. Wolfenstein, loc. cit. [6], p. 529.

Translated by I. Emin

NATURE OF LOW-TEMPERATURE MAGNETIC ANOMALIES IN FERRITES WITH COMPENSATION POINTS

K. P. BELOV

Moscow State University

Submitted to JETP editor April 6, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 692-695 (September, 1961)

By consideration of the nature of the temperature dependence of the spontaneous magnetization in gadolinium ferrite and in lithium ferrite-chromite, one can arrive at a condition for maximum probability of the appearance in ferrites of a point of compensation of the magnetic moments of the sublattices, or of an anomalous temperature variation of the spontaneous magnetization in the low-temperature region. The condition is this: one of the magnetic sublattices must possess "strong" exchange interaction, and consequently a Weiss temperature variation of spontaneous magnetization, and a second must possess a "weak" exchange interaction, which leads (in the presence of antiferromagnetic coupling to the "strong" sublattice) to a smeared out temperature variation of the spontaneous magnetization. It is concluded that in ferrites with compensation points, one should observe low-temperature magnetic anomalies, corresponding to an abrupt change of the long-range magnetic order in the sublattice with the "weak" exchange interaction. This conclusion is confirmed by the experimental data (cf. [2] and [4]).

1. In studies of ferrite-garnets of rare-earth elements^[1,2] and of lithium ferrite chromite^[3,4] in the low-temperature region (liquid nitrogen and liquid helium), anomalies have been observed in the temperature variation of magnetic and nonmagnetic properties (magnetic susceptibility, magnetostriction, coercive force, electrical resistivity, etc.).

The nature of these anomalies has hitherto not been explained. In this article we wish to show that the occurrence of the anomalies mentioned is connected with the existence of "low-temperature" points, corresponding to a sudden change of the long-range magnetic order in a sublattice with weak exchange interaction. Such a point should be especially well exhibited in ferrites with a compensation temperature, since in them there is a sharp difference in the nature of the exchange interactions in the sublattices; this in fact is the reason for the occurrence in the ferrite of a point of compensation of the magnetic moments of the sublattices.

2. We will analyze the condition for occurrence of a compensation point, for example, in the ferrite $3\text{Gd}_2\text{O}_3 \cdot 5\text{Fe}_2\text{O}_3$.

This ferrite is known to have three sublattices. On one of these (sublattice c) are located the Gd^{3+} ions, on the other two (a and d) the Fe^{3+} ions. The last two sublattices, experiment shows,

may be regarded as a single sublattice, thanks to the strong exchange interaction between the Fe^{3+} ions. We shall therefore refer hereafter to gadolinium ferrite-garnet as a ferrite consisting of two sublattices, "gadolinium" and "iron". We consider what form the $\sigma_s(T)$ curves will have for each of these sublattices.

In order to find the form of the $\sigma_s(T)$ curve for the iron sublattice, it is necessary to "switch off" the influence of the gadolinium sublattice in the magnetic sense, i.e., to replace all the Gd^{3+} ions with nonmagnetic ions, for example Y^{3+} or Lu^{3+} ions. Such a "substituted" ferrite is $3\text{Y}_2\text{O}_3 \cdot 5\text{Fe}_2\text{O}_3$ or $3\text{Lu}_2\text{O}_3 \cdot 5\text{Fe}_2\text{O}_3$. These ferrites, experiment shows, have a normal Weiss temperature variation of the spontaneous magnetization (Fig. 1, curve 1). Complete destruction of the magnetic long-range order occurs in them at a high temperature ($\sim 560^\circ\text{K}$), and this indicates the presence in the iron sublattice of strong exchange interaction. Knowing the approximate form of the temperature dependence of the spontaneous magnetization of the iron sublattice, we can determine, from the experimental $\sigma_s(T)$ curve for gadolinium ferrite (Fig. 1, curve 3), the approximate form of the curve for the gadolinium sublattice (Fig. 1, curve 2). It is evident that the gadolinium sublattice has an extremely spread-out temperature variation of the spontaneous magnetization, for which it is difficult

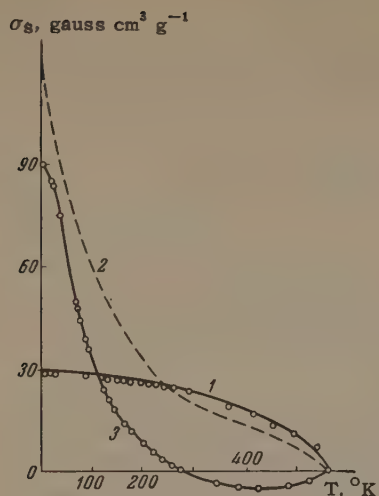


FIG. 1

even to define the position of the Curie point. Several alloys (for example 30% Cu, 70% Ni and others) have a similar type of $\sigma_s(T)$ curve.

One asks, why is the gadolinium sublattice distinguished by an extremely diffuse type of $\sigma_s(T)$ curve? Obviously the diffuseness, in the case of gadolinium ferrite-garnet, cannot be explained by fluctuations of the concentrations of magnetic and nonmagnetic ions over the volume of the ferrite, such as occur for example in the case of the alloy 30% Cu, 70% Ni.^[5] The fact is that the ferrite-garnets are distinguished by a strictly ordered distribution of ions in the lattice. In our opinion the diffuse form of the $\sigma_s(T)$ curve for the gadolinium sublattice is explained by the fact that the establishment of long-range magnetic order in this sublattice depends on two interactions: the positive exchange interaction between Gd^{3+} and Gd^{3+} ions (in sublattice c) and the negative exchange interaction between Gd^{3+} and Fe^{3+} ions (action of the iron sublattice upon sublattice c).

The effect of the negative exchange interaction on the long-range magnetic order in sublattice c will be greater, the smaller the positive exchange interaction within sublattice c. Therefore if the positive exchange interaction is "switched off" on increase of temperature, the long-range magnetic order, though it decreases abruptly, does not completely disappear, since the negative exchange interaction delays its destruction. This also explains the diffuse character of the $\sigma_s(T)$ curve of the gadolinium sublattice. The negative exchange interaction in gadolinium ferrite is of appreciable magnitude, as is indicated by the comparatively high value of the compensation temperature (without negative exchange interaction between the sublattices, occurrence of a compensation point would be impossible).

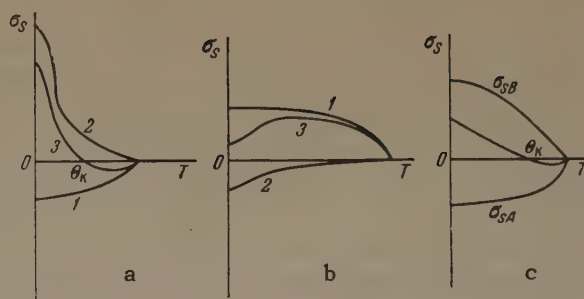


FIG. 2

Everything that has been said leads to a condition under which, with maximum probability, the occurrence of a compensation point in ferrites is to be expected. This condition is that in the ferrite one of the sublattices must have strong exchange interaction, and consequently a Weiss type of $\sigma_s(T)$ curve (Fig. 2a, curve 1), and another sublattice must have "weak" exchange interaction, which leads (in the presence of antiferromagnetic interaction from the "strong" sublattice) to a spread-out temperature variation of $\sigma_s(T)$ (Fig. 2a, curve 2). The compensation point θ_C occurs as a result of the subtraction from curve 2 (corresponding to the sublattice with the greater magnetic moment at 0° K) of curve 1 (with the smaller magnetic moment at 0° K). If the "weak" sublattice has a smaller magnetic moment at 0° K (Fig. 2b, curve 2) than has the "strong" (Fig. 2b, curve 1), then the subtraction of the curves results in an anomalous temperature variation of $\sigma_s(T)$ of the ferrite at low temperatures (curve 3, curve type M or P of Néel^[6]). Similar curves were observed in yttrium ferrite with part of the Y^{3+} ions replaced by ions Tb^{3+} , Gd^{3+} , etc. As a special case, Néel's type L curve can be obtained if the magnetic moments of the "weak" and "strong" sublattices at 0° K are equal (curve 3 emerges from 0° K), i.e., the compensation point is located in the immediate vicinity of 0° K.

3. We now consider a lithium ferrite-chromite, for example of composition $\text{Li}_{0.5}\text{Fe}_{2.5-a}\text{Cr}_a\text{O}_4$ (with $a = 1.25$), which according to Gorter^[7] possesses a compensation point. In this ferrite there are two antiferromagnetically interacting sublattices; on the first of these are located Fe^{3+} , Cr^{3+} , and Li^{1+} ions (octahedral sites), on the second Fe^{3+} and Li^{1+} ions (tetrahedral sites). At 0° K, according to a calculation by Néel's method, the magnetic moment of the first sublattice is equal to $5.45 \mu_B$, of the second $4.55 \mu_B$. In the low-temperature region, the ferromagnetism of the ferrite is determined by the octahedral sublattice. Indirect experimental evidence suggests that there is weaker exchange interaction in the octahedral sublattice

than in the tetrahedral. Such evidence is provided by measurements of the magnetic properties of chromium ferrites, which, the measurements have shown, possess low Curie points. Furthermore, in the presence of antiferromagnetic interaction between the octahedral and tetrahedral sublattices, the former must have a spread-out type of $\sigma_S(T)$ curve. Here also, therefore, conditions for the appearance of compensation points emerge.

We point out that according to Néel,^[6] a $\sigma_S(T)$ curve with a compensation point can be obtained also from $\sigma_{SB}(T)$ and $\sigma_{SA}(T)$ curves (for the octahedral and tetrahedral sublattices) of the Weiss type, provided the sublattice magnetizations σ_{SB} and σ_{SA} (near 0° K) are nearly equal. Then in consequence of the presence of very small irregularities in the temperature variation on the Weiss curves (cf. Fig. 2c), appearance of a compensation point is possible here also. However, such a case is seldom realized in practice. Furthermore it follows from Fig. 2 that the compensation point can occur only at temperatures close to the Curie point, whereas in real ferrites it is usually observed at a distance from the Curie point.

4. Everything that has been said above leads to the interesting conclusion that in ferrites with a compensation temperature, there should occur a "low-temperature" point, corresponding to an abrupt change of the magnetic long-range order in the sublattice with the "weak" negative exchange interaction. This point should be accompanied by anomalies in the temperature variation of magnetic and other physical characteristics of the ferrite.

Ped'ko^[2] has studied in detail, for gadolinium ferrite-garnet at liquid-nitrogen temperature, the anomalies of the temperature dependence of the paraprocess susceptibility, the magnetostriction, the coercive force, the width of the resonance

curve, and the resonance field; and he has shown that they are of a type reminiscent of those ordinarily observed at the Curie point. Goryaga and Lin Chang-ta,^[4] in experiments with lithium ferrite-chromite of various compositions, showed that if a compensation point occurs in the ferrite, then there appear in it at the same time, in the low-temperature region, anomalies in the temperature variation of the spontaneous magnetization, coercive force, magnetic susceptibility, galvanometric effect, and electrical resistivity. The nature of these anomalies suggests that in the low-temperature region a "Curie point," as it were, of the weak sublattice is manifesting itself. However, from the point of view of the thermodynamics of phase transitions one should, strictly speaking, not use the term "Curie point" here, since the change of long-range order in the "weak" sublattice comes about in the presence of the effective field of the "strong" sublattice.

¹R. Pauthenet, *Ann. phys.* **3**, 424 (1958); K. P. Belov, M. A. Zaitseva, and A. V. Ped'ko, *JETP* **36**, 1672 (1959), *Soviet Phys. JETP* **9**, 1191 (1959).

²A. V. Ped'ko, *JETP* **41**, 700 (1961), this issue p. 505.

³E. W. Lee and R. R. Birss, *Proc. Phys. Soc. (London)* **76**, 411 (1960).

⁴A. H. Goryaga and Lin Chang-ta, *JETP* **41**, 696 (1961), this issue p. 502.

⁵K. P. Belov, *Magnitnye prevrashcheniya* (Magnetic Transformations), Fizmatgiz, 1959.

⁶L. Néel, *Ann. phys.* **3**, 137 (1948).

⁷E. W. Gorter, *Philips Res. Reports* **9**, 295 (1954).

LOW TEMPERATURE MAGNETIC ANOMALIES IN LITHIUM FERRITE CHROMITES

A. N. GORYAGA and LIN CHANG-TA

Moscow State University

Submitted to JETP editor April 6, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 696-699 (September, 1961)

The temperature dependences of spontaneous magnetization, coercive force, magnetic susceptibility, and electrical resistance were measured for lithium ferrite chromites which have compensation points of the magnetic moments of the sublattices. The anomalies found in the temperature variation of such magnetic and electrical characteristics at low temperatures in these ferrites confirm Belov's suggestion^[1] that ferrites possessing compensation points should display low-temperature magnetic anomalies that correspond to a sharp change in the magnetic order in a sublattice with "weak" exchange interaction.

1. Lithium ferrite chromites are of note in that in some the magnetic moments of the sublattices have compensation temperatures, Θ_{comp} . Investigations^[1,2] made earlier established that the physical properties of the ferrites change sharply on going through the temperature Θ_{comp} . The reason for this is that the magnetic properties, and the physical properties of the ferrite associated with them, are determined in the region below Θ_{comp} by an octahedral sublattice and above Θ_{comp} by a tetrahedral sublattice. A study of ferrites with a compensation point is therefore of great interest to the theory of ferrimagnetism.

The work cited above was mainly devoted to a study of the physical properties of lithium ferrite chromites in the region of the compensation point. In the present work we have undertaken the measurement of the magnetic and of some electrical characteristics of lithium ferrite chromites at low temperatures ($T < \Theta_{\text{comp}}$).

2. The following ferrites were prepared by ceramic methods for the investigation: $\text{Li}_2\text{O} \cdot 3\text{Fe}_2\text{O}_3 \cdot 2\text{Cr}_2\text{O}_3$ ($\Theta_{\text{comp}} = 523.5^\circ\text{K}$) $\text{Li}_2\text{O} \cdot 2.5\text{Fe}_2\text{O}_3 \cdot 2.5\text{Cr}_2\text{O}_3$ ($\Theta_{\text{comp}} = 326^\circ\text{K}$) and $\text{Li}_2\text{O} \cdot 5\text{Fe}_2\text{O}_3$ (without a compensation point). A preliminary annealing was carried out for four hours at 1000°C in air. The final sintering was made for three hours at 1200°C in air, followed by slow cooling. The magnetization σ , the magnetic susceptibility χ in a field $H = 10$ oe, and the coercive force H_c were measured by a ballistic method in a solenoid. All the specimens were parallelepipeds with pointed ends.

The electrical resistance R was measured on a dc Wheatstone bridge. In view of the fact that the electrical resistance of ferrites at the tem-

peratures of interest to us is very high, we chose specimens in the shape of thin plates 1 mm thick and 9×9 mm in area. Electrical contacts were produced on the specimens by baking a silver paste at a temperature of 500°C . In order to obtain a continuous range of temperatures—from liquid-nitrogen to room temperature—the specimen was contained in a special molybdenum glass oven, placed in a dewar containing liquid nitrogen.

3. As a result of the investigations it was established that in ferrites with a compensation point Θ_{comp} , the temperature dependence of spontaneous magnetization σ_s has a noticeable break in the region of liquid nitrogen temperature. Corresponding to this, minima and maxima are observed on the curves of H_c and χ vs. temperature.

The temperature dependences of σ_s , H_c and χ for the ferrites $\text{Li}_2\text{O} \cdot 3\text{Fe}_2\text{O}_3 \cdot 2\text{Cr}_2\text{O}_3$ and Li_2O

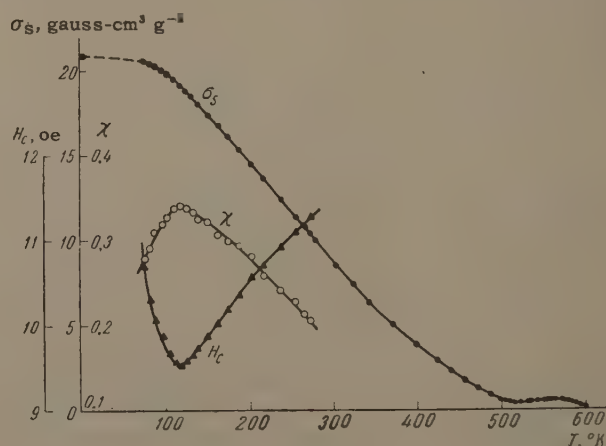


FIG. 1. The temperature dependences of spontaneous magnetization σ_s , coercive force H_c and susceptibility χ (at $H = 10$ oe) of the ferrite $\text{Li}_2\text{O} \cdot 3\text{Fe}_2\text{O}_3 \cdot 2\text{Cr}_2\text{O}_3$ ($\Theta_{\text{comp}} = 523.5^\circ\text{K}$).

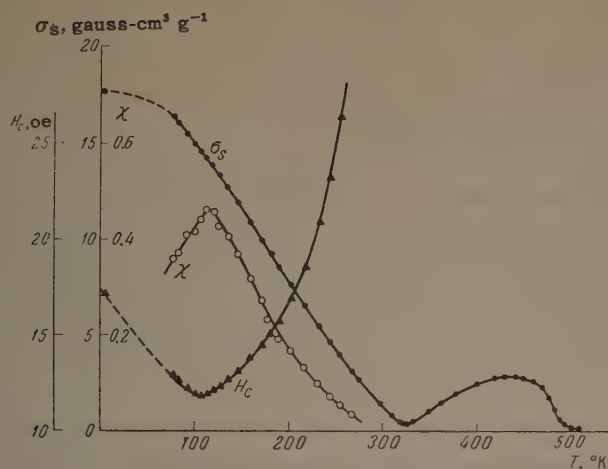


FIG. 2. The temperature dependences of the spontaneous magnetization σ_s , H_c and χ of the ferrite $\text{Li}_2\text{O} \cdot 2.5\text{Fe}_2\text{O}_3 \cdot 2.5\text{Cr}_2\text{O}_3$ ($\Theta_{\text{comp}} = 326^\circ\text{K}$).

$\cdot 2.5\text{Fe}_2\text{O}_3 \cdot 2.5\text{Cr}_2\text{O}_3$ are shown in Figs. 1 and 2. It can be seen that the magnetization σ_s of both ferrites changes little with increasing temperature up to liquid-nitrogen temperatures, but then decreases rapidly (showing a break) up to the compensation temperature. The susceptibility χ has a maximum where the spontaneous magnetization starts to fall rapidly. The coercive force has a minimum at this temperature and then increases rapidly as the temperature increases.

Analogous behavior of the temperature curves of σ_s , H_c and χ is observed for all compositions of lithium ferrite chromites which have a compensation point. In lithium ferrite chromites without a compensation point, the $\sigma_s(T)$, $H_c(T)$ and $\chi(T)$ curves do not have low temperature anomalies. This can be seen in Fig. 3, which gives measurement data on the ferrite $\text{Li}_2\text{O} \cdot 5\text{Fe}_2\text{O}_3$; here σ_s and H_c decrease smoothly while χ increases. We also measured the temperature dependence of the electrical resistance R for lithium ferrite chromites with a Θ_{comp} point. We established that at the temperatures at which anomalies

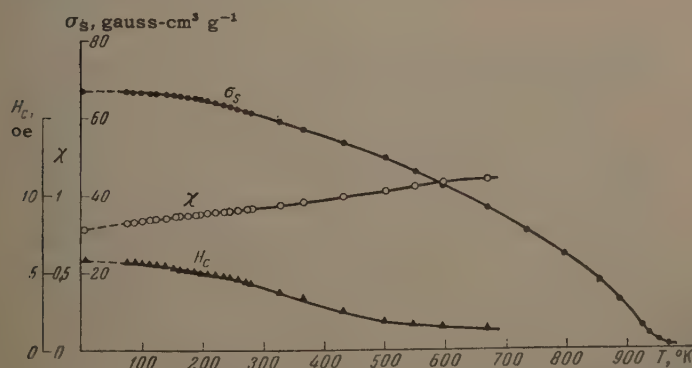
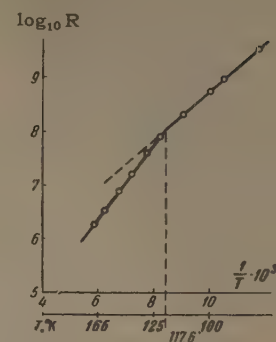


FIG. 3. The temperature dependences of σ_s , H_c and χ of the ferrite $\text{Li}_2\text{O} \cdot 5\text{Fe}_2\text{O}_3$.

FIG. 4. The temperature dependence of electrical resistance of the ferrite $\text{Li}_2\text{O} \cdot 2.5\text{Fe}_2\text{O}_3 \cdot 2.5\text{Cr}_2\text{O}_3$.



in the temperature variation of σ_s , H_c and χ are observed, the $\log_{10} R (1/T)$ curve shows a break (Fig. 4). It can be seen from Figs. 1, 2, and 4 that the anomalies in the temperature variation of σ_s , H_c , χ and R are analogous to those observed in ferromagnets near the Curie point.

4. The anomalies appearing in the temperature variation of the magnetic and electric characteristics at low temperatures are related to the occurrence of a rapid change in the long range magnetic order in the ferrites studied. According to Belov,^[3] such a change must take place in ferrites which have compensation points in a sublattice with "weak" exchange interaction. The existence of such a "weak" sublattice in a ferrite is a condition for the appearance of a compensation point. In lithium ferrite chromites the octahedral sublattice is the "weak" sublattice, in which there is a considerable amount of chromium and lithium ions as well as iron ions. The presence of the lithium leads to a great weakening of the exchange interaction.

In the tetrahedral sublattice, on the other hand, there is a stronger exchange interaction, since there are many Fe ions in it (and a negligible amount of Li ions). Since there is a strong negative exchange interaction between the sub-

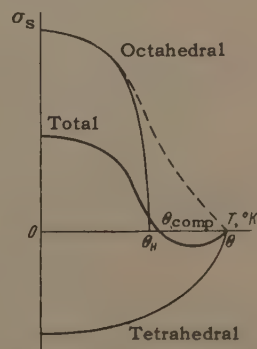


FIG. 5. Schematic curves of the temperature dependence of the spontaneous magnetization of each sublattice separately and the total magnetization of a ferrite; Θ is the Curie point, Θ_{comp} the compensation point and Θ_H the temperature at which a sharp change in magnetic long range order takes place.

lattices, it can exert a considerable influence on the magnetic ordering in them, especially in the "weak" sublattice. As a result of this, the temperature dependence of the spontaneous magnetization of the sublattice with weak interaction can have a spread out appearance. In other words, the long range magnetic order in the weak sublattice does not completely disappear as a result of the influence of the negative exchange interaction between the sublattices.

The temperature dependence of the spontaneous magnetization of each sublattice separately and of the ferrite as a whole is shown schematically in Fig. 5. The dashed curve also shows the $\sigma_s(T)$ curve for the octahedral sublattice, spread out as a result of the influence of an antiferromagnetic interaction. The influence of the negative exchange interaction apparently shows up most strongly in that part of the $\sigma_s(T)$ curve where a partial destruction of long range order has already taken place, i.e., the spontaneous magnetization has decreased strongly compared with the value of σ_s of the other sublattice. The $\sigma_s(T)$ curve for the octahedral sublattice will, therefore, have a long "tail" up to the Curie temperature Θ of the tetrahedral sublattice. The $\sigma_s(T)$ curve for the octahedral sublattice is not, consequently, of

the Weiss type. According to Belov,^[3] the presence of such a "tail" in the spontaneous magnetization is the cause of the appearance of a compensation point Θ_{comp} .

The reason for the anomalies observed by us in the temperature dependence of σ_s , H_C , χ and R at low temperatures ($T < \Theta_{\text{comp}}$) is thus related to the fact that in these ferrites the octahedral sublattice shows a weak positive exchange interaction, as a result of which the long range magnetic order, produced by the interaction, is destroyed at low temperatures.

In conclusion the authors thank Professor K. P. Belov for his valuable advice during the carrying out of this work.

¹Belov, Goryaga, and Lin Chang-ta, JETP **38**, 1914 (1960), Soviet Phys. JETP **11**, 1376 (1960).

²K. M. Bol'shova and T. A. Elkina, Fiz. Metal. metalloved (Physics of Metals and Metallography) **8**, 461 (1959). K. P. Belov and A. V. Ped'ko, JETP **39**, 961 (1960), Soviet Phys. JETP **12**, 666 (1961).

³K. P. Belov, JETP **41**, 692 (1961), this issue, p. 499.

ANOMALIES IN THE PHYSICAL PROPERTIES OF GADOLINIUM GARNET FERRITE IN THE LOW TEMPERATURE REGION

A. V. PED'KO

Moscow State University

Submitted to JETP editor April 6, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 700-702 (September, 1961)

The temperature dependences of the magnetic and magnetoelastic properties and of the ferromagnetic resonance parameters of gadolinium garnet ferrite have been measured at low temperatures. It was established that the curves of the temperature variation of the initial susceptibility and of the paraprocess susceptibility have maxima, while minima are found in the curves of coercive force and of the width of the resonance curve. There are also maxima of the paraprocess magnetostriction and of the effect of uniform pressure on the spontaneous magnetization at this temperature. These results confirm Belov's suggestion^[2] that low temperature anomalies, corresponding to a sharp change in the magnetic long range order in a sublattice with "weak" exchange interaction, should occur in ferrites possessing compensation points.

1. Measurements of the temperature dependence of the magnetostriction of gadolinium garnet ferrites^[1] showed that the magnetostriction isotherms at temperatures above the compensation point Θ_{comp} are of the form usual for "normal" ferromagnets ($\lambda_{\parallel}$ and $\lambda_{\perp}$ are of opposite sign and $\lambda(H)$ exhibits saturation in the isotherms). However, on reducing the temperature below Θ_{comp} a relatively large volume magnetostriction is superimposed on this magnetostriction, as a result of which saturation is not observed in the isotherms, even in strong fields. The slope of the high field parts of the curves increases with decreasing temperature and reaches a maximum in the region of liquid nitrogen temperature. This behavior of the magnetostriction curves resembles that usually observed on approaching a ferromagnetic Curie point.

From an analysis of the results of measurements on the temperature dependence of the spontaneous magnetization of ferrites with a compensation point, Belov^[2] deduced that anomalies should appear at low temperatures in such ferrites, corresponding to a rapid change in the long range magnetic order in the sublattice with "weak" exchange interaction (in our case, in the gadolinium sublattice).

2. In order to confirm this conclusion we undertook measurements of the temperature dependence of a number of physical characteristics of gadolinium ferrite in the low temperature range.

It can be seen from Fig. 1 that a maximum in the magnetic permeability μ_0 , measured in a weak field ($H = 1.75$ oe), is found at a temperature of 100° K. It can be seen that the coercive force H_C has a minimum in the region of 100° K. The susceptibility due to the paraprocess, χ_p , of gadolinium garnet ferrite was measured in fields from 1,000 to 3,000 oe in the temperature range from 80° K to room temperature. The maximum in the paraprocess susceptibility can also be seen to be at 100° K.

Measurements of the temperature dependence of the width of the resonance absorption curve, ΔH , of gadolinium ferrite were made at 9,400 Mc/sec, by the short circuited waveguide section method, in the temperature range 4 to 560° K. These measurements showed that ΔH is very great at liquid helium temperatures (Fig. 1), then decreases with increasing temperature, reaches a minimum in the temperature region of 90 — 100° K and then increases again with increasing temperature. We should also mention that the magnitude of the resonance field H_r decreases sharply on cooling below 100° K.

The form of the anomalies near 100° K in the temperature dependence of all the properties measured are thus similar to those usually observed in ferromagnets near the Curie point. This indicates that near 100° K a rapid change in the magnetic long range order takes place in the gadolinium sublattice, in agreement with Belov's conclusions.^[2]

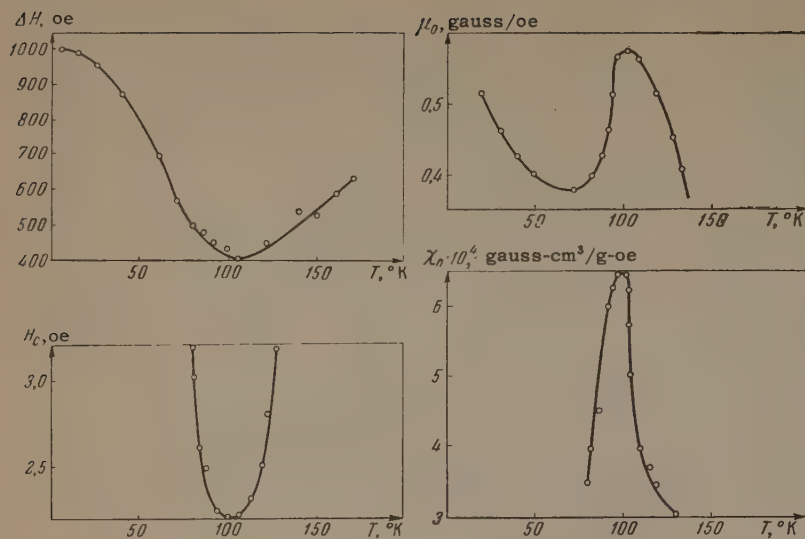


FIG. 1. Anomalies in the temperature variation of some physical properties of gadolinium garnet ferrite at low temperatures.

3. The existence of a relatively large volume magnetostriction by the paraprocess in gadolinium garnet ferrite (in the region of the low temperature magnetic transition) leads one to assume that the exchange interaction between Gd^{3+} ions in the gadolinium sublattice depends strongly on the interatomic distance. Uniform elastic deformations should then exert an appreciable influence on the magnetization of gadolinium ferrite near this transition. We made measurements of the magnetization of gadolinium ferrite at 80° K. The specimen was contained in a bomb in which water was frozen, thus being subjected to a hydrostatic compression on the order of 1,800 atm.

It can be seen from Fig. 2 that the magnetization decreases with increasing pressure, which agree with the sign of the magnetostriction. It follows from simple thermodynamic considerations that the change of volume in the magnetization process at constant pressure (i.e., the volume magnetostriction) is related to the change in magnetization in a field under the action of pressure, by the relation

$$(\partial V / \partial H)_p / V = - (\partial I / \partial p)_H.$$

Our measurements gave

$$(\partial I / \partial p)_H \approx \rho (\Delta \sigma / \Delta p)_H = 8 \cdot 10^{-9} \text{ oe}^{-1}$$

(assuming that the density ρ changes little with pressure). From the measurements of magnetostriction^[1] we obtained

$$(\partial V / \partial H)_T \approx 3 \Delta \lambda / \Delta H = 3 \cdot 10^{-9} \text{ oe}^{-1},$$

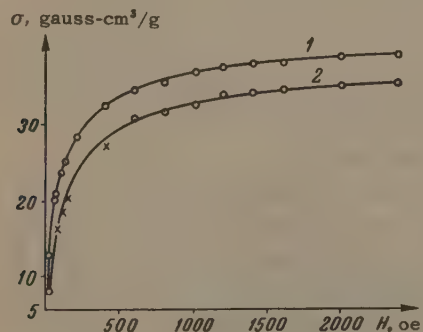


FIG. 2. Magnetization isotherms $\sigma(H)$ of gadolinium garnet ferrite, measured at 80° K: curve 1 — at atmospheric pressure, curve 2 — at a pressure of 1,800 atm.

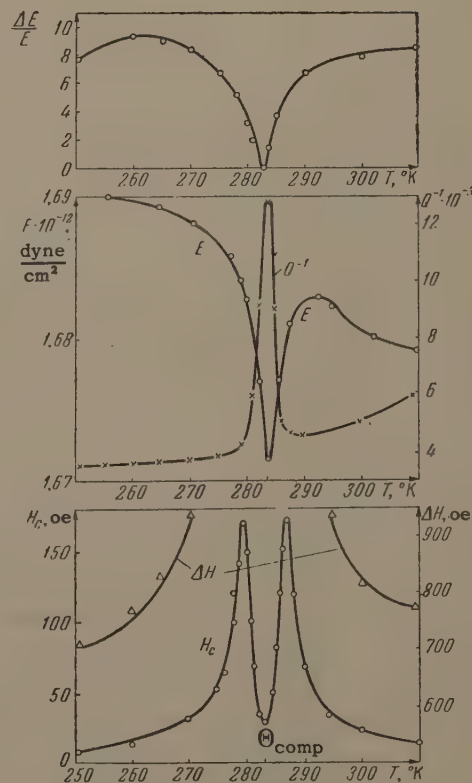


FIG. 3. Anomalies in the temperature variation of some physical properties of gadolinium garnet ferrite at the compensation point Θ_{comp} .

i.e., a value of the same order of magnitude.

4. In conclusion, we show that our measurements have established the existence of sharp anomalies in various magnetic and non-magnetic properties of gadolinium ferrite at the compensation point. It can be seen from Fig. 3 that at Θ_{comp} the coercive force, H_c , and the width of the resonance absorption curve, ΔH , have maxima.^[3,4] The internal friction Q^{-1} , measured at $H = 1,800$ oe, also has a maximum at the compensation point, while Young's modulus E has a sharp minimum. The ratio $\Delta E/E$ (the ΔE -effect) equals zero at the compensation point. We shall explain the physical nature of these anomalies elsewhere.

In conclusion, the author expresses his thanks to Professor K. P. Belov for his interest in this work and for discussion of the results.

¹K. P. Belov and A. V. Pedko, J. Appl. Phys. Suppl. **31**, 55 (1960).

²K. P. Belov, JETP **41**, 692 (1961), this issue, p. 499.

³K. P. Belov and A. V. Ped'ko, JETP **39**, 961 (1960), Soviet Phys. JETP **12**, 666 (1961).

⁴Smith, Overmeyer, and Calhoun, IBM J. Res. Developm. **3**, 153 (1959).

Translated by R. Berman

126

ALPHA-PARTICLE SPECTRA AND DIFFERENTIAL CROSS SECTIONS FOR THE REACTION $H^3(t, 2n)He^4$ AT AN ANGLE OF 90°

A. M. GOVOROV, LI HA YOUN, G. M. OSETINSKII, V. I. SALATSKII, and I. V. SIZOV

Joint Institute for Nuclear Research

Submitted to JETP editor April 21, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 703-707 (September, 1961)

Spectra of α particles emitted from the $H^3(t, 2n)He^4$ reaction at a laboratory angle of 90° were measured with a proportional counter and a 50-channel pulse-height analyzer. The bombarding tritium nuclei had energies ranging between 230 and 1000 kev. The differential cross section for the reaction was calculated on the basis of the spectra thus obtained and found to increase monotonically from 3 to 5.2 mb/sr as the energy increased. At all incident energies there was an appreciable contribution from the two-stage process involving the formation of a neutron and a He^5 nucleus which subsequently decayed into an α particle and a neutron.

INTRODUCTION

A number of papers have been devoted to studying the T-T reaction and, in particular, the energy spectrum of the α particles produced.

Allen et al.^[1-3] have measured the energy spectra of α particles emitted at 90° for bombarding tritium energies of 220–240 kev. A continuous energy distribution of α particles extending up to 3.8 Mev was observed, with a broad maximum in the (2.7–2.9)-Mev region; the shape of the maximum bore a good resemblance to that of the spectrum calculated under the assumption that two neutrons and an α particle were simultaneously produced in the reaction. The contribution from the process corresponding to the emission of two virtually bound neutrons (a "dineutron") with a lifetime greater than 10^{-21} seconds was estimated to be less than 1% of the total yield.

Using a 2.5-Mev electrostatic generator, the Los Alamos group^[4] has investigated the neutron yield and α -particle spectra at various angles. Preliminary indications of the existence of the dineutron were obtained.

Jarmie and Allen^[5] have measured α -particle spectra at a laboratory angle of 30° for incident tritium energies of $E_T = 0.95 - 2.10$ Mev, and at 30° , 60° , 90° , and 120° for $E_T = 1.9$ Mev. From the analysis it was concluded that these spectra could be explained approximately by the theory of a two-stage process proceeding through the formation of He^5 and a neutron with the subsequent decay of the He^5 into an α particle and a neutron. Since the conclusions of ^[1-3] and ^[5] differ considerably as regards the mechanism of the T-T reaction, it was of interest to trace the change in character of the α -particle spectrum in the unin-

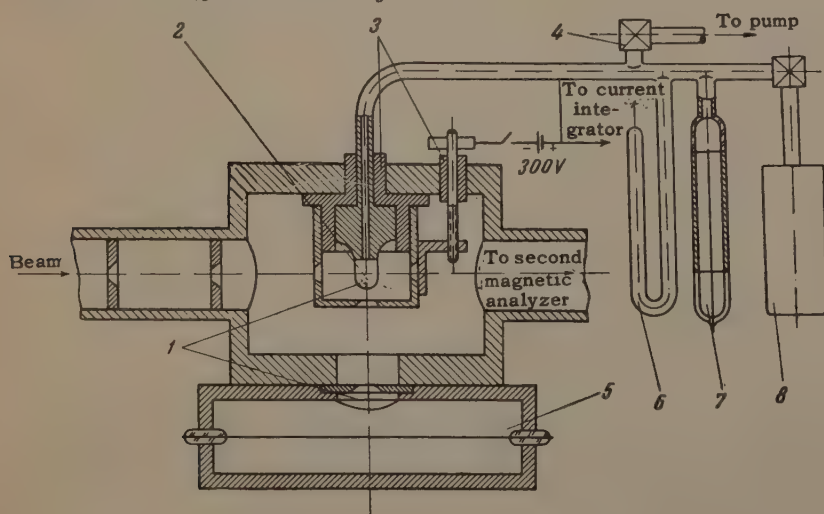


FIG. 1. Schematic diagram of the apparatus. 1 – Mica foil, 2 – gas target, 3 – insulator, 4 – valve, 5 – proportional counter, 6 – mercury manometer, 7 – ionization chamber, 8 – cylinder containing uranium tritide.

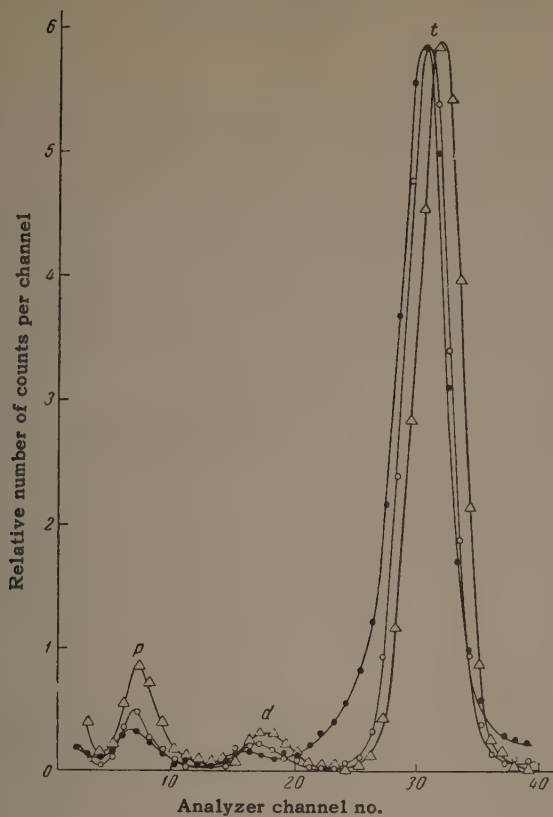


FIG. 2. Spectra of particles from the scattering of the mass 3 beam on argon. Δ — is for a target current of 0.005 μ amp, O — of 0.05 μ amp, and $\bullet$ — of 0.14 μ amp.

vestigated (200–1000)-keV range of particle bombarding energies.

EXPERIMENTAL METHOD

After passing through the magnetic analyzer, the triton beam from the electrostatic generator

struck a thin gas target (cf. Fig. 1) surrounded by an insulated metal housing which collected the charge of the beam. A proportional counter (diameter 80 mm, length 220 mm) filled at a pressure of 5 atm with pure argon (containing a few percent CO_2) was placed at 90° to the beam. The entrance and exit windows of the target were made out of a single piece of mica foil 0.10–0.14 mg/cm² thick. The counter window was closed by a 0.18 mg/cm² mica foil. The counter had an energy resolution of 4.5–5% for Pu^{239} α particles. After amplification, pulses from the proportional counter were fed into the 50-channel pulse-height analyzer.

The target was filled with tritium (extracted from uranium tritide) at 60 mm mercury. The concentration of tritium in the target, measured with an ionization chamber calibrated by tritium of known concentration, was about 70%. All vacuum seals of the target and filling system were made of teflon and metal. The composition of the beam of mass 3 ions was determined by scattering it on argon at 90° and measuring the energy spectra of the scattered particles. Figure 2 presents the spectra obtained at the triton bombarding energy of 820 keV. The highest peak corresponds to the tritons, the next one to deuterons from ionized DH^+ molecules, and the third peak to protons obtained from the scattering of molecular HHH^+ and DH^+ ions. According to estimates which took into account the nuclear reaction cross section and multiple Coulomb scattering, the deuterium content of the beam of tritons did not exceed 0.4%, and the HHH^+ ion content was always less than 0.1%.

The same deuterium contamination in the beam was

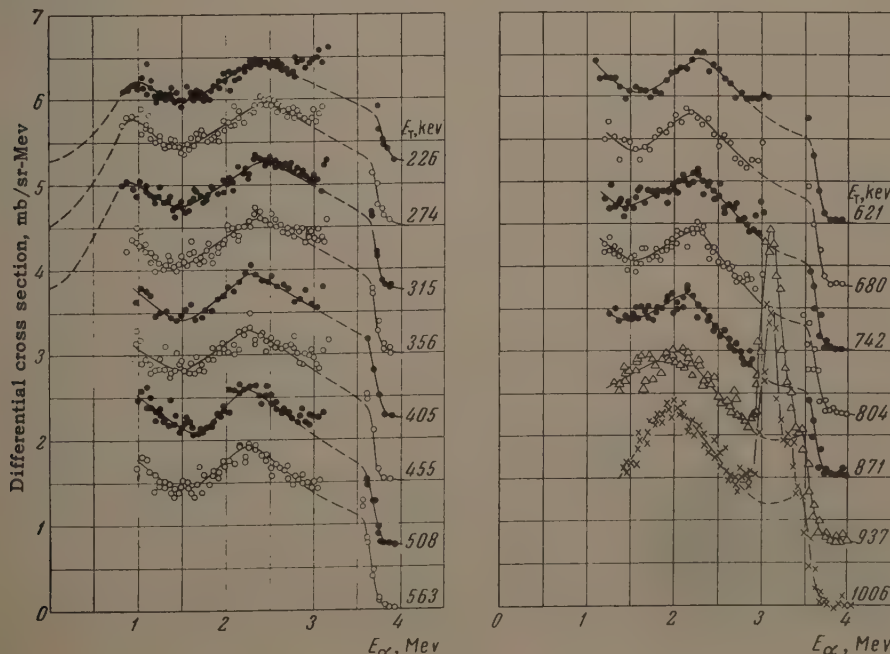


FIG. 3. Spectra of α particles from the T-T reaction at an angle of $\theta_{lab} = 90^\circ$ for various tritium nucleus energies. Consecutive curves are displaced 0.75 mb/sr-MeV along the vertical.

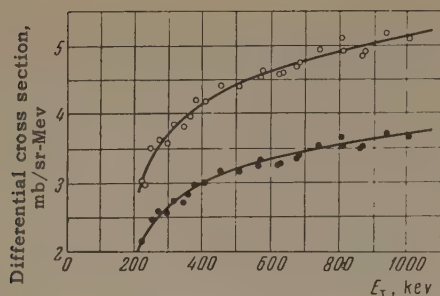


FIG. 4. Cross section for the T-T reaction at a laboratory angle of 90° integrated over α -particle energies; O — is the cross section integrated over the whole α -particle spectrum, ● — is the contribution from 61% of the α -particle spectrum (according to energy scale).

also found by evaluating the contribution of the $H^3(d, n)He^4$ reaction to the T-T reaction yield.

The energy losses of tritons in the target were determined with the aid of a second magnetic analyzer placed in the path of the beam behind the target.

TREATMENT AND DISCUSSION OF THE RESULTS

The experimentally obtained pulse-height distributions were transformed into energy distributions of α particles by using the range-energy curve for α particles in mica.^[6]

The final α -particle spectra for various triton bombarding energies are presented in Fig. 3. Peaks from the D-T reaction caused by the presence of deuterium in both target and beam were found in the right-hand part of each spectrum. In order not to complicate the drawing, these peaks are shown in only two spectra, for $E_T = 937$ and 1006 kev. That part of the spectrum of α particles from the T-T reaction which lies under these peaks was obtained by subtracting the contribution from the D-T reaction, whose character was determined separately by bombarding deuterium with tritons and tritium with deuterium.

The region of low α -particle energies could not be investigated satisfactorily because of the background of tritons scattered on the slight nitrogen, oxygen, and mercury contaminants in the target. The latter could not be eliminated even after the necessary precautionary measures were taken. Therefore the extrapolation of the spectra to zero energy was most justified for $E_T = 226, 274$, and 315 kev.

The extrapolation of the α -particle spectrum to zero energy adopted was based, first, on the theoretically predictable spectrum shape calculated assuming a three-particle decay of He^6 and a two-stage process involving the formation of He^5 , and

second, on spectrum measurements obtained by Jarmie and Allen.^[5]

A wide maximum in the 2.2-Mev region and a minimum in the 1.5-Mev region are very evident in all spectra, and in the majority of the spectra the rise of the curve to a second maximum (in the region of low α -particle energies) is distinctly observed. Control experiments on the scattering of tritons by the gas residues in the target (after absorption of the tritium by uranium) showed that this rise cannot be explained by scattered tritons and is due to the T-T reaction.

The shape of the spectra (position of maxima and minima) indicates that the two-stage process involving the formation of a neutron and He^5 (in the ground state) with subsequent rapid decay of He^5 into an α particle and neutron plays an appreciable role at triton bombarding energies in the 200 — 1000 kev range.

At large incident triton energies a group of α particles appears at the right-hand end of the spectra whose presence can be explained as the correlation of two neutrons in the three-particle breakup of the compound nucleus. This group of α particles is obscured at low triton energies by the large peak from the D-T reaction; the shape of the right-hand side of the spectra can only be approximated here. Errors of up to 30% may occur in this part of the spectrum, while in the middle section the mean-square error of each point at all triton bombarding energies does not exceed 4 — 7%. Comparatively larger errors were considered likely in the extrapolation of spectra to zero α -particle energy.

Figure 4 presents differential cross sections for the T-T reaction integrated over α -particle energies. Since at $E_T = 1006$ kev only a part of the α particle spectrum was recorded (encompassing about 61% of the energy scale, counting from the right-hand end of the spectrum), these same parts were separated out on all the spectra and the corresponding areas obtained for them. In addition, the integration was carried out over the whole spectrum in the case of the first three spectra ($E_T = 226, 274$, and 315 kev), where a reasonable extrapolation to zero energy was possible. Assuming that the ratio of the total area of the spectrum to the area separated out of its right-hand part does not depend on the triton energy, differential cross sections were obtained which were integrated over all α -particle energies for the remaining energies E_T also. Although the aforementioned assumption can not claim to be strict, the error in determining the differential

cross sections (according to the estimates made) does not exceed 20%.

In conclusion the authors express deep gratitude to F. L. Shapiro for his valuable advice, and to the collaborators from the electrostatic generator group for their aid in carrying out the experiments.

¹ Allen, Almqvist, Dewan, Pepper, and Sanders, Phys. Rev. **82**, 262 (1951).

² Allen, Almqvist, Dewan, and Pepper, Phys. Rev. **81**, 315 (1951).

³ Allen, Almqvist, Dewan, Pepper, and Sanders, Phys. Rev. **79**, 238 (1950).

⁴ Los Alamos Scientific Laboratory, Phys. Rev. **79**, 238 (1950).

⁵ N. Jarmie and R. C. Allen, Phys. Rev. **111**, 1121 (1958).

⁶ Govorov, Li Ha Youn, Osetinskii, Salatskii, and Sizov, Preprint, Joint Institute for Nuclear Research, R-725.

Translated by Mrs. J. D. Ullman
127

EXPERIMENTAL VERIFICATION OF THE CHARGE INDEPENDENCE PRINCIPLE IN THE $d + d \rightarrow \text{He}^4 + \pi^0$ REACTION FOR 400-Mev DEUTERONS

Yu. K. AKIMOV, O. V. SAVCHENKO, and L. M. SOROKO

Joint Institute for Nuclear Research

Submitted to JETP editor April 21, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 708-724 (September, 1961)

Experiments with a 400-Mev deuteron beam are described in which the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction, forbidden by the law of conservation of total isotopic spin, is investigated. It is shown that the total cross section for the reaction is less than $1.1 \times 10^{-32} \text{ cm}^2$ on the 90% confidence level. The upper limit of the cross section is compared with the cross section of the $d + d \rightarrow \text{He}^4 + \gamma$ reaction and with the cross section of the $p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+$ reaction, which is allowed by the isotopic spin selection rules. The data obtained in the present study, together with the theoretical estimates, indicate that the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction is strongly forbidden and hence confirm the validity of the law of conservation of total isotopic spin in processes of pion production by nucleons and light nuclei.

1. INTRODUCTION

THE law of conservation of total isotopic spin is the most important consequence of the principle of charge independence and leads to fully determined relations between the cross sections for different processes and to the formulation of additional selection rules. In the present study, we consider only one of the possible classes of phenomena in which the principle of charge independence leads to important consequences, namely, pion production by nucleons and light nuclei. In this case it is possible to verify the charge independence principle in the following ways:

A. It is known that the total cross sections for various processes of the form

$$N + N \rightarrow N + N + \pi \quad (1)$$

satisfy the relation

$$\sigma(pp \rightarrow pn\pi^+) + \sigma(pn \rightarrow nn\pi^+) + \sigma(pn \rightarrow pp\pi^-) = 2[\sigma(pp \rightarrow pp\pi^0) + \sigma(pn \rightarrow pn\pi^0)]. \quad (2)$$

This relation has been verified experimentally for incident nucleon energies from the pion production threshold to 660 Mev.^[1] At 600 Mev, where the cross sections were measured with the smallest error, relation (2) is satisfied to an accuracy of 8–10%.

B. Between the differential cross sections of the two charge-conjugate reactions

$$p + p \rightarrow d + \pi^+, \quad (3)$$

$$n + p \rightarrow d + \pi^0 \quad (4)$$

the following relation holds:

$$d\sigma(p + p \rightarrow d + \pi^+) = 2d\sigma(n + p \rightarrow d + \pi^0). \quad (5)$$

Verification of this relation at 400 Mev^[2] and 590 Mev^[3] showed that no significant deviation is observed. A stricter verification of relation (5) requires, along with an increase in the accuracy of the measurements, the consideration of a number of perturbation corrections resulting from the nonidentical properties of different charge states of the nucleon and pion as well as the Coulomb forces in reaction (3).

C. Verification of the relations between differential cross sections of the reactions

$$p + d \rightarrow \text{H}^3 + \pi^+, \quad (6)$$

$$p + d \rightarrow \text{He}^3 + \pi^0, \quad (7)$$

representing two independent channels from the same initial state. This method has certain advantages from the experimental viewpoint, since it permits the elimination of a number of systematic errors. Calculation of the Clebsch-Gordan coefficients^[4] for the $(\text{H}^3\pi^+)$ and $(\text{He}^3\pi^0)$ states leads to the relation

$$d\sigma(p + d \rightarrow \text{H}^3 + \pi^+) = 2d\sigma(p + d \rightarrow \text{He}^3 + \pi^0). \quad (8)$$

But as in the case of reactions (3) and (4), a perturbation effect arises here, too, owing to the nonidentical properties of the neighboring charge states of the H^3 and He^3 nuclei and of the π^0 and π^+ mesons. The size of the possible perturbations in reactions (6) and (7) were calculated for a proton energy of 600 Mev.^[5] It turned out that these per-

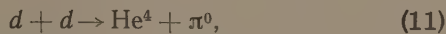
turbations alter the relation between the cross sections for processes (6) and (7), so that, instead of (8), we have

$$d\sigma(p + d \rightarrow H^3 + \pi^+) = 2(1.10 \pm 0.035) d\sigma(p + d \rightarrow He^3 + \pi^0). \quad (9)$$

The limited accuracy of similar calculations places a limit of sensitivity on this method of verifying relation (9). Reactions (6) and (7) have been studied at 370 Mev^[6] and 450 Mev.^[7] The most accurate results were obtained at 600 Mev^[8] and gave

$$d\sigma(p + d \rightarrow H^3 + \pi^+) = 2(1.065 \pm 0.03) d\sigma(p + d \rightarrow He^3 + \pi^0). \quad (10)$$

D. The study of the forbidden reaction

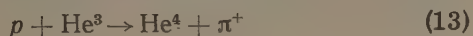


as suggested by Lapidus,^[9] occupies a special place. Reaction (11) involves particles in only one charge state, and therefore the perturbation resulting from the difference between neighboring charge states is automatically absent. The study of reaction (11) also attracts interest, since it allows a strict verification of the hypothesis that there exists a π_0^0 meson^[10] with a scalar isotopic spin.

We present below new data on reaction (11), which was studied with 400-Mev deuterons. The first results on the study of this reaction, obtained by us earlier, have already been published [see^[11] and also the proceedings of the conferences on high-energy physics at Kiev (1959) and Rochester (1960)]. Furthermore, we report the results of the cross section measurements for the electromagnetic process



at the same deuteron energy and for the reaction



produced by 670-Mev protons. We also give theoretical estimates of the cross section for pion production by nucleons and light nuclei, which make it possible to draw additional conclusions on the degree of forbiddenness of reaction (11).

2. EXPERIMENTAL ARRANGEMENT

The method of experimental verification of the charge independence principle by the measurement of the degree of forbiddenness of process (11) has the advantage that by formulating the problem in this way we do not require accurate measurements of the total or differential cross sections, as is the case for the comparison of two charge-conjugate

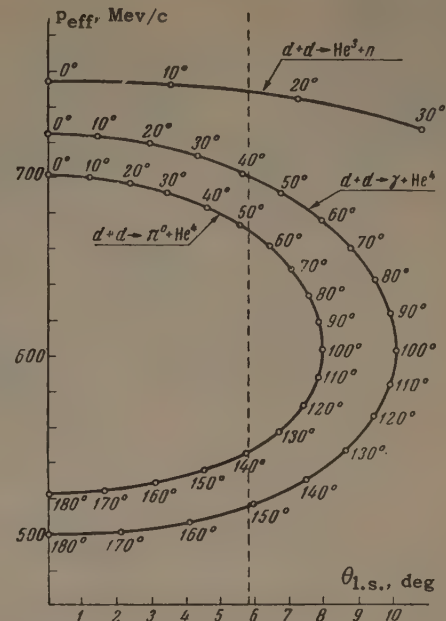


FIG. 1. Kinematics of the reactions $d + d \rightarrow He^4 + \pi^0$, $d + d \rightarrow He^4 + \gamma$, and $d + d \rightarrow He^3 + n$ for an incident deuteron energy of 404 Mev (θ is the angle of emission of the He^4 and He^3 nuclei; P_{eff} is the effective momentum of these particles and is equal to p/Z , where Z is the charge of the nucleus). The numbers along the curves are the angles of emission of the He^4 and He^3 nuclei in the c.m.s. of the colliding deuterons. The dotted line corresponds to the angle at which the heavy particles were recorded, i.e., 5.8° l.s.

reactions. The experimentally determined cross section for the forbidden process or the estimate of its upper limit is a measure of the violation of the forbiddenness. This method is essentially a null method of measurement. We can draw conclusions as to the degree of forbiddenness of reaction (11) if, in addition to this reaction, we study the electromagnetic process (12) and reaction (13); the latter is allowed by the selection rules resulting from the law of conservation of total isotopic spin.

The kinematics of reactions (11), (12), and (13) for 404-Mev deuterons is shown in Fig. 1. The limiting angle of emission of the α particle is 8° in reaction (11) and 10° in reaction (12) in the laboratory system (l.s.). The π^0 energy in the center-of-mass system (c.m.s.) is 82 Mev, which allows us to assume that the π^0 meson is emitted primarily in the s and p states. This conclusion follows from the fact that occurrence of d waves in reaction (3) becomes appreciable in the polarization effects beginning with a π^+ -meson energy of ~ 100 Mev,^[12] and the effect of the d wave on the angular distribution as a result of the $\cos^4 \theta$ terms is not significant even for very high energies.^[13] On the other hand, the emission of π^0 mesons in reaction (11) in the s and p states is not forbidden by any additional selection rules,

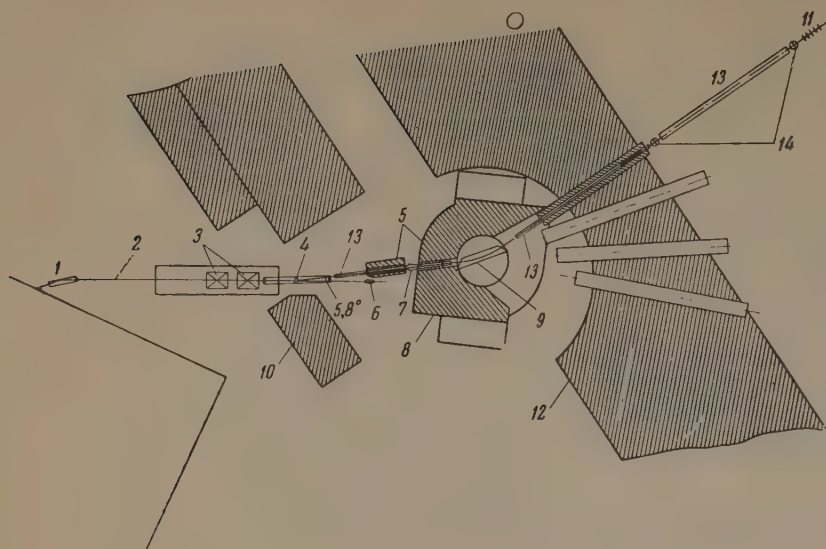


FIG. 2. Experimental setup: 1 — beam deflector; 2 — deuteron or proton beam; 3 — magnetic quadrupole lenses; 4 — gaseous or liquid deuterium target or carbon target; 5 — lead shield; 6 — monitor; 7 — trajectory of secondary charged particles; 8 — bending electromagnet; 9 — focusing shims; 10 — concrete shield; 11 — telescope consisting of five scintillation counters; 12 — shielding wall; 13 — evacuated tube; 14 — scintillation counters for separating particles on basis of time of flight.

apart from the law of conservation of total isotopic spin, and in the case of the violation of the latter, the differential cross section of reaction (11) has the form $A + B \sin^2 \theta$. The measurements of reaction (11) were therefore carried out for an isotropic angle (55° in the c.m.s. or 5.8° in the l.s.).

The basic transition in reaction (12) is the electric quadrupole transition $^1S \rightarrow ^1D$,^[14] which is responsible for an angular distribution of the form $\sin^2 \theta \cos^2 \theta$.* The c.m.s. angle of observation of the α particles is 41° , i.e., quite close to the maximum of the differential cross section.

Measurements of the He^3 yield from the reaction



for a l.s. angle of emission 5.8° were used for the determination of the resolving power of the recording apparatus, for resetting the apparatus, and for control measurements. The efficiency of the apparatus was checked by comparison of reactions (6) and (7) produced by 670-Mev protons. The absolute cross sections for all the foregoing processes were determined from measurements of the deuteron yield from reaction (3), whose cross section is well known over a wide energy range.^[13,15]

3. CONDITIONS OF THE EXPERIMENT

The experimental setup is shown in Fig. 2. The deuteron or proton beam was focused on the target by means of a two-sector magnetic quadrupole lens with an 80-mm aperture. The secondary charged particles produced in the target were separated by a brass collimator set at an angle of 5.8° with re-

spect to the beam axis and were deflected in the magnetic field by an angle 27° , after which they passed through a steel collimator in a concrete shielding wall and were recorded by a system of scintillation counters.

At the exit of the synchrocyclotron chamber, the deuteron beam had a mean energy of 405.3 ± 0.5 Mev and a dispersion of 1.7 ± 0.5 Mev.^[16] The deuteron energy at the target dropped to 404 Mev. Photographic paper was exposed at different points along the deuteron trajectory to aid in the adjustment of the beam and the control of its shape. The total intensity of the deuteron beam was $\sim 0.6 \times 10^{11} \text{ sec}^{-1}$. The beam was monitored by an emission chamber located behind the target. We used 220-Mev protons to calibrate the emission chamber for the deuteron beam by comparison of the currents from the emission and the ionization chambers. In this way, we were able to find the cross section of reaction (14), which was used later on for the estimate of the cross sections of processes (11) and (12).

4. RECORDING APPARATUS

The total cross section for dd collisions is $\sim 10^{-24} \text{ cm}^2$ and hence for the measurement of cross sections less than 10^{-31} cm^2 the recording apparatus should possess a degree of selection so as to separate one "favorable" event from a background of several million other particles. The charged particles from reactions (3), (6), (7), (11) — (14) were identified by the effective momentum p/Z (Z is the charge of the particle), specific ionization, and range; in later experiments, selection was also made on the basis of the time of flight.

*The angular distribution of reaction (12) reported in the Proceedings of the Tenth Annual Conference on High Energy Physics at Rochester (p. 51) is in error.

The secondary charged particles with a given effective momentum were separated by an electromagnet with a pole diameter of 100 cm and gap of 13 cm. Iron shims of trapezoidal shape and thickness 3.2 cm were placed between the poles of the magnet. The lateral bevel of the shims provided horizontal focusing of the particles at the point of location of the apparatus.

To record charged particles of high ionization and small range in a background of other radiation of lower ionization we used the method of separating particles on the basis of their ionization in several scintillation counters.^[17] This method makes it possible to decrease appreciably the contribution from particles of low ionization produced in the "tail" in the ionization spectrum and also to suppress the background caused by star-formation processes in the scintillator material and by random superposition of pulses in the electronic equipment.

The telescope designed for the selection of particles by this method consisted of six scintillation counters. The first five counters, connected in coincidence, were spectrometric counters, and the sixth counter was connected in anticoincidence with the first five. Connected in this way, the apparatus selected only those events in which all counters, apart from the anticoincidence counter, gave simultaneous pulses whose amplitudes at the output of the amplifiers exceeded the discrimination threshold. In the selection of particles on the basis of their range, aluminum and copper filters were placed between counters 4 and 5 of the telescope. The differential range interval was fixed by a filter placed before the anticoincidence counter. In the later series of measurements, an additional selection of the particles was made on the basis of the time of flight. The pulses from the first two counters, 3.3 m apart, were applied to a coincidence circuit with a resolving time of $(5 - 7) \times 10^{-9}$ sec.

5. ADJUSTMENT OF THE APPARATUS

The basic characteristics of the recording system were determined with a beam of α particles of different energy^[17] and by the identification of heavy charged particles which are produced in reactions (6), (7), and (14) with comparatively large cross section ($\gtrsim 10^{-29}$ cm²). The overall efficiency of the apparatus was checked by comparison of the yields of H^3 and He^3 nuclei produced in reactions (6) and (7). Reaction (6) was studied earlier^[18] with a 670-Mev proton beam. The He^3 yield from reaction (7) was measured with a deu-

terium gas target filled to a pressure of 3.5 atm. The experimental arrangement and the method of selection of the particles were the same as those described in^[18].

To determine the absolute cross section of reaction (7) under the same conditions, we measured the deuteron yield from reaction (3). It was found that

$$\frac{d\sigma}{d\Omega}(\theta_{He^3} = 5.8^\circ) = (20.1 \pm 1.8) \cdot 10^{-30} \text{ cm}^2/\text{sr} \quad (15)$$

while the c.m.s. differential cross section with respect to the angle of emission of the π^0 meson turned out to be

$$\frac{d\sigma^*}{d\Omega^*}(\theta_{\pi^0}^* = 12^\circ) = (4.5 \pm 0.4) \cdot 10^{-30} \text{ cm}^2/\text{sr} \quad (16)$$

The triton yield from reaction (6) was measured a second time with an additional selection made on the basis of the time of flight. The results obtained in this way were in agreement with the first series of measurements^[18] within the limits of experimental error.

These measurements made it possible to find the ratio of the cross sections of reactions (6) and (7) for a π^0 c.m.s. angle $\sim 12^\circ$:

$$\begin{aligned} d\sigma(p + d \rightarrow H^3 + \pi^+) \\ = (2.04 \pm 0.32) \cdot d\sigma(p + d \rightarrow He^3 + \pi^0). \end{aligned} \quad (17)$$

The obtained ratio of cross sections was in quite good agreement with the calculated value 2.20 ± 0.07 ,^[5] within the limits of accuracy of the measurements. By comparing our results with more accurate measurements,^[8] we can conclude that the recording apparatus had practically the same efficiency of registration for both singly and doubly charged particles.

6. THE $d + d \rightarrow He^3 + n$ REACTION

Before each series of measurements with the deuteron beam, we checked the apparatus by measuring the He^3 yield from reaction (14). The He^3 particles were identified by the momentum, ionization, range, and time of flight (in the last series of measurements). The comparatively large cross section for this reaction ($\sim 10^{-28}$ cm²) made it possible to measure the He^3 yield quite reliably and comparatively quickly. The conditions of recording the He^3 nuclei during measurements with different targets and with different methods of selection are shown in Figs. 3, 4, and 5.

The differential cross section of reaction (14) for a l.s. angle 5.8° was determined independently in several runs by relating it to the cross section of reaction (3) when the synchrocyclotron was switched from deuteron acceleration to proton ac-

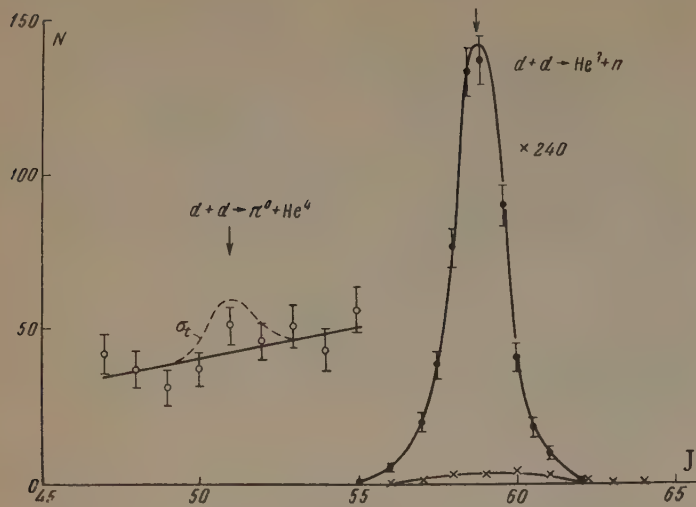


FIG. 3. Conditions of recording α particles from the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction and He^3 nuclei from the $d + d \rightarrow \text{He}^3 + n$ reaction with a liquid deuterium target. O — background readings of the recording apparatus set for α particles from the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction as a function of the magnet current J; the dotted curve corresponds to a possible cross section of $2 \times 10^{-32} \text{ cm}^2$ for this reactions; ● — He^3 yield from the filled target; x — He^3 yield from the empty target. Here and in the following figures N and J are reduced to relative units (readings of the individual instruments).

celeration. The differential cross section of reaction (14) averaged over four runs without corrections for the absorption and multiple scattering in the copper decelerating filters of the telescope turned out to be

$$\eta \frac{d\sigma}{d\Omega} (5.8^\circ) = (2.18 \pm 0.15) \cdot 10^{-28} \text{ cm}^2/\text{sr} \quad (18)$$

The value of the correction η was measured by comparison of the He^3 from the liquid deuterium target with and without the filter. Taking into account this correction, we obtain

$$\frac{d\sigma}{d\Omega} (5.8^\circ) = (3.3 \pm 0.23) \cdot 10^{-28} \text{ cm}^2/\text{sr}, \quad (19)$$

or, in the c.m.s.,

$$\frac{d\sigma^*}{d\Omega^*} (16^\circ) = (4.3 \pm 0.3) \cdot 10^{-29} \text{ cm}^2/\text{sr}. \quad (20)$$

7. MEASUREMENTS WITH A LIQUID DEUTERIUM TARGET

The first data on reaction (11) were obtained with deuterium-polyethylene and carbon targets.^[11] It was found that the upper limit of the cross section for reaction (11) was

$$\sigma_t(d + d \rightarrow \text{He}^4 + \pi^0) < 1.0 \cdot 10^{-31} \text{ cm}^2 \quad (21)$$

on the 90% confidence level. An appreciable yield of α particles from collisions between deuterons and carbon nuclei did not permit a significant improvement in the accuracy of the results by a direct

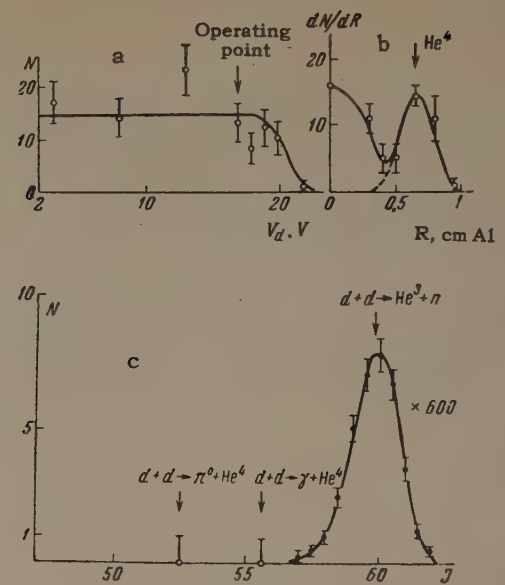


FIG. 4. Conditions of recording α particles from the $d + d \rightarrow \text{He}^4 + \pi^0$ and $d + d \rightarrow \text{He}^4 + \gamma$ reactions; a and b — counting responses of the recording apparatus set for α particles from the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction in measurements with a heavy polyethylene target (V_d — discrimination threshold, R — thickness of the decelerating filter in the telescope); c — results of the yield measurements from the $d + d \rightarrow \text{He}^3 + n$, $d + d \rightarrow \text{He}^4 + \pi^0$, and $d + d \rightarrow \text{He}^4 + \gamma$ reactions with a deuterium gas target.

increase in the duration of the measurements. The use of a liquid deuterium target made it possible to reduce the α -particle background to approximately $1/35$ of the background for a heavy polyethylene target.

The particles were identified by their effective momentum, ionization, and range as in the case of the first series of measurements.^[11] Reaction (14)

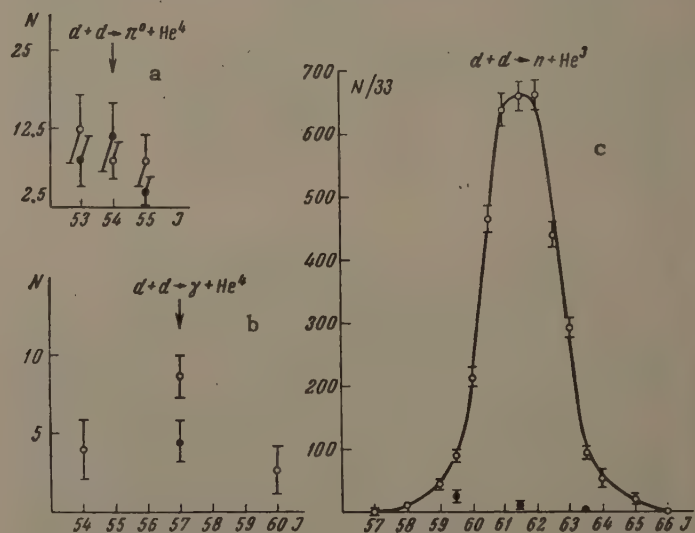


FIG. 5. Yield from the $d + d \rightarrow \text{He}^3 + n$, $d + d \rightarrow \text{He}^4 + \pi^0$, and $d + d \rightarrow \text{He}^4 + \gamma$ reactions as a function of the magnet current. O — measurements with a deuterium gas target, ● — measurements with a hydrogen target.

Reaction	$d + d \rightarrow \text{He}^3 + n$	$d + d \rightarrow \text{He}^4 + \gamma$	$d + d \rightarrow \text{He}^4 + \pi^0$
Effective momentum p/Z at the center of the deflecting magnet, Mev/c	743	689	656
Spread of the effective momentum due to the target thickness, Mev/c	0.6	5	6
Bending-magnet current, arbitrary units	60	55.6	52.5
Thickness of filter in the telescope	10.7 g/cm ² Cu	1.0 cm Al	0.75 cm Al
Thickness of filter in front of the anticoincidence counter	1.7 g/cm ² Cu	0.3 cm Al	0.3 cm Al
Mean ionization loss of particles dE/dx in carbon, Mev/g-cm ²	24	44	49
Time of flight of particles over 3.35-m base, 10 ⁻⁹ sec	24	33	35
Time of flight of protons of the same effective momentum, 10 ⁻⁹ sec	17	18	19
Initial energy of heavy particles for l.s. angle of 5.8°, Mev	374	254	232
Final energy of heavy particles after passing through five scintillation counters of the telescope, Mev	343	207	180

was used to adjust the apparatus. The considerable reduction of the background made it possible to obtain a better separation of the monoenergetic group of He^3 nuclei with the aid of all the counting responses (Fig. 3). The conditions of recording the α particles were similar to the conditions in the first series of measurements.^[11] The constancy of the difference in readings of the apparatus with two different filters indicated that, with a discrimination threshold above 18 v, practically only single α particles were recorded. These α particles were emitted under the action of deuterons from the brass foil of the deuterium target container. The He^3 yield from reaction (14) is shown in Fig. 3 in the scale 1:240 for a l.s. angle 5.8°. Also shown there are the background readings for bending-magnet currents corresponding to α particles from reaction (11). The straight line determined by the method of least squares for the experimental points located to the right and left of the calculated point for the α -particle peak from reaction (11) corresponds to the mean background level.

Comparison of the results of the measurements with this line makes it possible to estimate the upper limit of the cross section for the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction. If we exclude the change in the momentum interval in the transition from one bending-magnet current to another ($k_1 = 1.15$) and introduce a correction for the absorption of α particles in the decelerating filter ($k_2 = 1.10$) and for the scattering in air ($k_3 = 1.16$), then

$$\frac{d\sigma}{d\Omega}(5.8^\circ) < 1.35 \cdot 10^{-31} \text{ cm}^2/\text{sr} \quad (22)$$

on the 90% confidence level, or, in the c.m.s.,

$$\frac{d\sigma^*}{d\Omega^*}(55^\circ) < 1.5 \cdot 10^{-33} \text{ cm}^2/\text{sr} \quad (23)$$

If the π^0 angular distribution has the form $A + B \sin^2 \theta$, then

$$\sigma_t(d + d \rightarrow \text{He}^4 + \pi^0) < 1.9 \cdot 10^{-32} \text{ cm}^2 \quad (24)$$

on the 90% confidence level. By way of illustration, the dotted line in Fig. 3 corresponds to a possible cross section of $2 \times 10^{-32} \text{ cm}^2$ for this reaction. The estimate (24) also applies to the upper limit of the cross section for the production of the π_0^0 meson with a scalar isotopic spin and a rest mass of 100–150 Mev.

We give below some kinematical characteristics of α particles for different possible masses of the π_0^0 meson:

Possible values of the π_0^0 -meson mass, Mev	190	176	134.8	98	0
Corresponding effective momentum of the α particles, Mev/c	575	596	635	652	669
Bending-magnet current, arbitr. units	46.5	48.3	51.5	52.6	54.5
Thickness of the decelerating filter in the telescope, cm Al	0.265	0.384	0.62	0.71	0.87

8. MEASUREMENTS WITH A DEUTERIUM GAS TARGET

A further decrease in the background of α particles knocked out of the brass foil of the target container and also of fragments produced in the decelerating filters and scintillation counters was obtained with a deuterium gas target and the introduction of the additional selection of particles on the basis of the time of flight. Such a target consisted of a copper tube of 12.5-cm diameter and 150 cm long. The target was set in the beam in such a way as to prevent particles produced in the front and rear windows of the target from entering the collimator located in front of the bending magnet. The front window of 9 cm diameter was covered by a brass foil 200 μ thick. The exit flange of the target had two openings covered by a terylene film 200 μ thick. The deuterium gas was

supplied to the target from an arrangement containing activated carbon which was cooled to the temperature of liquid nitrogen. This procedure prevented, to a considerable degree, heavier gases from reaching the target. With a collimator entrance diameter of 40 mm and various positions of the recording apparatus, such a gas target is equivalent to a liquid deuterium target 2 to 3 mm thick.

The air was evacuated over almost the entire path of the particles from the target to the counters.

The table gives the most important kinematical characteristics of the recorded heavy particles at a l.s. angle 5.8° for 404-Mev deuterons.

The counting response of the apparatus set for α particles from reaction (11) was determined from measurements of α particles from the heavy polyethylene target (Figs. 4a, b). The stability of the apparatus during the measurements of α particles from reactions (11) and (12) was checked periodically by putting the heavy polyethylene target in place of the deuterium gas target. The quantity checked was the α -particle yield from this target. During the measurements with the deuterium gas target, no pulses were recorded from α particles which could have been produced in reactions (11) and (12). No change in the efficiency of the apparatus was observed. The results are shown in Fig. 4c in which the He^3 yield from reaction (14) is shown in a scale 1:600; α particles from reactions (11) and (12) are also shown. These data, together with the corrections mentioned above, permit us to determine the upper limit of the cross section for reaction (11):

$$\sigma_t(d + d \rightarrow \text{He}^4 + \pi^0) < 1.9 \cdot 10^{-32} \text{ cm}^2 \quad (25)$$

on the 90% confidence level.

Under these conditions, a further increase in the accuracy of the measurements can be attained by increasing the statistics of the measurements. For this reason we increased the dimensions of the scintillators, as a result of which the angle subtended by the recording apparatus rose approximately sevenfold without an appreciable worsening of the resolving power for the effective momentum. The remaining conditions of the experiment were the same as in the previous series of measurements.

Figure 5 shows the results of measurements under these changed conditions. On the right is the calibration peak for the He^3 yield from reaction (14). The occurrence of the background is due to the worsening of the transmission and to an increase in the efficiency for the recording of

fragments from stars. The background was determined with a hydrogen gas target filled to the same pressure as in the case of the deuterium target. It took two hours to change the target. During this time we measured α particles from carbon in order to check the stability of the apparatus. Over a period of time totaling about 20 hours and for three values of the bending-magnet current, 14 counts from the deuterium target and 13 counts from the hydrogen target were recorded for the same number of monitor counts. The mean number of counts in the recording of α particles from carbon was 17 ± 2 after the measurements with the deuterium target and 14 ± 1.2 after the measurements with the hydrogen target. For the evaluation of the data, we analyzed the time distribution of the counts. The observed distribution was in good agreement with a Poisson distribution and this was sufficient grounds to consider only the statistical errors in the evaluation of the data. After the introduction of the corrections, the results of this series of measurements gave

$$\sigma_t(d + d \rightarrow \text{He}^4 + \pi^0) < 1.8 \cdot 10^{-32} \text{ cm}^2 \quad (26)$$

on the 90% confidence level.

In the measurement of α particles from reaction (12) we recorded 37 counts during a period of about 20 hours. The background under these conditions was determined in two ways: first, by measurements with a deuterium target for small and large bending magnet currents and second, with a hydrogen target. During about 20 hours of measurements, we recorded a total of 17 background counts with the same number of monitor counts. The results of the measurements of the efficiency of the apparatus obtained by putting a carbon target in place of the gas target indicated that the mean number of

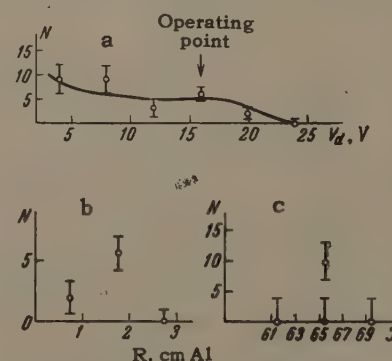


FIG. 6. Conditions of recording α particles from the $p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+$ reaction. a, b — counting response of the recording apparatus set for α particles from the $p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+$ reaction obtained with a carbon target; c — results of the measurement of the yield for the $p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+$ reaction as a function of the magnet current: O — measurements with an He^3 gas target; ● — measurements with a deuterium gas target.

α -particle counts was 18 ± 1.3 after the measurements with a deuterium target and 20 ± 1.4 after measurements with a hydrogen target. As before, we analyzed the time distribution of the counts. No deviation from a Poisson distribution was observed. The difference between the counts of the apparatus at the position of the calculated peak for α particles from reaction (12) and the background level was 20 ± 7.4 . The probability that this difference actually equals zero is less than 1%. This is sufficient cause to consider the observed positive effect from reaction (12) to be quite real. If we take into account small corrections, then the differential cross section for the $d + d \rightarrow \text{He}^4 + \gamma$ reaction at 5.8° l.s. is

$$\frac{d\sigma}{d\Omega} (5.8^\circ) = (0.84 \pm 0.31) \cdot 10^{-31} \text{ cm}^2/\text{sr}. \quad (27)$$

The c.m.s. differential cross section is

$$\frac{d\sigma^*}{d\Omega^*} (41.5^\circ) = (1.6 \pm 0.6) \cdot 10^{-33} \text{ cm}^2/\text{sr}. \quad (28)$$

If we take into account the angular distribution of reaction (12), $d\sigma/d\Omega \sim \sin^2 \theta \cos^2 \theta$, then the total cross section is

$$\sigma_t (d + d \rightarrow \text{He}^4 + \gamma) = (1.1 \pm 0.4) \cdot 10^{-32} \text{ cm}^2. \quad (29)$$

9. THE $p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+$ REACTION

We measured the cross section for this reaction, which is allowed by the isotropic spin selection rules, with a 670-Mev proton beam at 5.8° l.s. The experimental setup, position of the apparatus, and the selection methods employed were the same as for the measurements described in Sec. 8. Some of the kinematical characteristics of the high-energy branch of α particles produced in reaction (13) were as follows:

Angle of recording α particles in the l.s.	5.75°
Angle of emission of the pion in the c.m.s.	159°
Effective momentum p/Z	802 Mev/c
α -particle energy	329 Mev
dE/dx	30 Mev/g·cm ⁻²
Time of flight over 3.35-m base	29.5×10^{-9} sec
Thickness of decelerating filter	1.76 cm Al
Thickness of filter in front of anticoincidence counter	0.8 cm Al

The apparatus was set for α particles from a carbon target of the same energy as the α particles from reaction (13) (Fig. 6). The basic measurements were made with a gas target filled with He^3 to a pressure of 0.9 atm. The background was measured with a deuterium gas target filled to a pressure of 3 atm. The stability of the apparatus was checked every two hours by measurements of the α -particle yield from the carbon target. During about seven hours of measurement 10 counts were recorded. When the bending-magnet current

was increased or decreased, no counts of the apparatus were recorded. Also, no counts were recorded from the deuterium target for a magnet current corresponding to α particles from reaction (13). The absolute cross section of reaction (13) was calibrated by means of the triton yield from reaction (6) found under the same conditions as with the deuterium gas target.

The differential cross section of reaction (13) for a l.s. angle 5.8° was

$$\frac{d\sigma}{d\Omega} (5.8^\circ) = \left(3.26 \begin{smallmatrix} +1.35 \\ -1.27 \end{smallmatrix} \right) \cdot 10^{-30} \text{ cm}^2/\text{sr} \quad (30)$$

and in the c.m.s.

$$\frac{d\sigma^*}{d\Omega^*} (\theta_\pi^* = 159^\circ) = \left(0.24 \begin{smallmatrix} +0.10 \\ -0.09 \end{smallmatrix} \right) \cdot 10^{-30} \text{ cm}^2/\text{sr}. \quad (31)$$

10. DISCUSSION OF RESULTS

A. Comparison of the $d + d \rightarrow \text{He}^4 + \pi^0$ and the $d + d \rightarrow \text{He}^4 + \gamma$ reactions. If we average all the estimates of the upper limit of the cross section for reaction (11) found in the three series of measurements (24), (25), and (26), then in the l.s. we have

$$\frac{d\sigma}{d\Omega} (5.8^\circ) < 8 \cdot 10^{-32} \text{ cm}^2/\text{sr}, \quad (32)$$

in the c.m.s.

$$\frac{d\sigma^*}{d\Omega^*} (55^\circ) < 9 \cdot 10^{-34} \text{ cm}^2/\text{sr} \quad (33)$$

and the total cross section is

$$\sigma_t (d + d \rightarrow \text{He}^4 + \pi^0) < 1.1 \cdot 10^{-32} \text{ cm}^2 \quad (34)$$

on the 90% confidence level. This value of the upper limit of the cross section of reaction (11) should be compared with the cross section of the electromagnetic process (12) which, on the basis of the data obtained in the present experiment, is $(1.1 \pm 0.4) \times 10^{-32} \text{ cm}^2$.

An independent estimate of this cross section can be obtained from the data on the photodisintegration of helium nuclei.^[19] According to the results of [19], the integral cross section of the electromagnetic process

$$\gamma + \text{He}^4 \rightarrow d + d, \quad (35)$$

defined as

$$\int_{E_\gamma=23 \text{ Mev}}^{190} \sigma_t(E_\gamma) dE_\gamma,$$

is $\sim 2\%$ of the integral cross section of the reaction

$$\gamma + \text{He}^4 \rightarrow \text{H}^3 + p. \quad (36)$$

The low yield for reaction (35) is explained by the

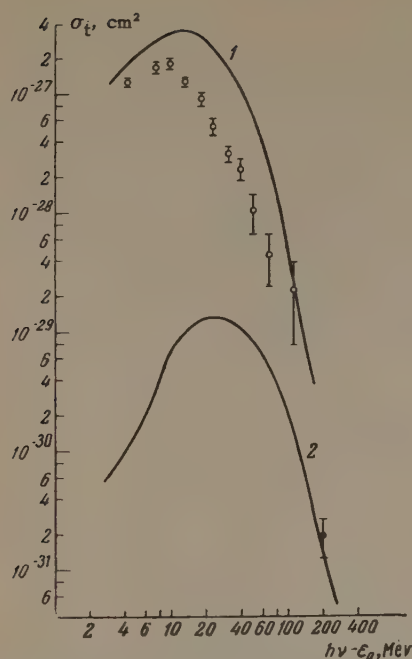


FIG. 7. Energy dependence of the reactions $\gamma + \text{He}^4 \rightarrow \text{H}^3 + \text{p}$ (curve 1) and $\gamma + \text{He}^4 \rightarrow \text{d} + \text{d}$ (curve 2); ϵ_0 — reaction threshold; O — experimental points of [19]; ● — data of the present experiment.

fact that the electric dipole transition in this reaction is forbidden and that the electric quadrupole transition $^1\text{S} \rightarrow ^1\text{D}$ is more probable. If it is assumed that the energy dependence of the total cross section of reaction (35) is of the same form as in the case of reaction (36), then the cross section of the inverse process (12) calculated with the aid of the principle of detailed balance is $\sim 0.8 \times 10^{-32} \text{ cm}^2$ for incident 400-Mev deuterons.

Gorbunov and Spiridonov [19] analyzed the energy dependence of the electric quadrupole transition in reaction (36). This analysis indicated that the relative contribution from the quadrupole transition should increase with the energy of the γ quanta. This conclusion qualitatively agrees with the theoretical estimates [14] made for processes (35) and (36). If, finally, we take into account the increase in the cross section of photoprocesses as a result of resonances in the meson cloud, which is most clearly observed in the photodisintegration of deuterons in the 200 — 300 Mev energy region, [20] then it becomes obvious that the estimate of $0.8 \times 10^{-32} \text{ cm}^2$ for the cross section of (12) is only a lower limit, and the total cross section can actually exceed this value.

The energy dependence of the cross section of reactions (35) and (36) can be considered theoretically, similarly to the treatment of Flowers and Mandl [14]. We carried out calculations with wave functions of the H^3 and He^4 nuclei taken in the form [21]

$$\Psi \sim e^{-\alpha R}, \quad R = \left(\sum_{i < j} r_{ij}^2 \right)^{1/2}, \quad (37)$$

where r_{ij} is the distance between nucleons in the nucleus. For the H^3 nucleus, the choice of the parameter α was based on the value of the energy of the Coulomb repulsion of the He^3 nucleus. [22] For the He^4 nucleus, the parameter α was found from the value of the rms radius of the α particle. [23] For the deuteron, we took the Hulthen function. The results of the calculation are shown in Fig. 7. Also shown in the figure are the experimental data for reaction (36) obtained in experiments on the photodisintegration of helium in the γ -quantum energy interval from the threshold to 190 Mev [19] and data for reaction (12) obtained in the present experiment and recalculated for the inverse reaction (35).

The calculated dependence of the cross section for reaction (36) is in qualitative agreement, within the limits of a factor of 2.5, with the experimental data in the entire energy interval. The very steep drop in the cross section in the high energy region is due to the fact that in the calculations only the electric dipole transition was taken into account, while from the data on the angular distribution of reaction (36) [19] it follows that the electric quadrupole transition begins to play an appreciable role starting with an energy of 30 Mev, and its contribution increases with the γ -quantum energy. The calculated integral cross section of reaction (35) is

$$\int_{E_\gamma=23 \text{ Mev}}^{220} \sigma_t(E_\gamma) dE_\gamma = 0.72 \cdot 10^{-27} \text{ cm}^2 \text{ Mev} \quad (38)$$

which is in good agreement with the value $0.75 \times 10^{-27} \text{ cm}^2 \text{ Mev}$ obtained from experiments on the photodisintegration of helium. [19] The calculated cross section of reaction (35) for 220-Mev γ quanta is $1.25 \times 10^{-31} \text{ cm}^2$, which, when recalculated for reaction (12), gives

$$\sigma_t(d + d \rightarrow \text{He}^4 + \gamma) = 0.7 \cdot 10^{-32} \text{ cm}^2 \quad (39)$$

The cross section of reaction (12) can thus be estimated in three ways: 1) experimentally ($1.1 \times 10^{-32} \text{ cm}^2$); 2) by extrapolation of the data on the cross section for the photodisintegration of helium ($0.8 \times 10^{-32} \text{ cm}^2$); 3) by direct calculation ($0.7 \times 10^{-32} \text{ cm}^2$). Taken together, the results give

$$\sigma_t(d + d \rightarrow \text{He}^4 + \gamma) \approx 1 \cdot 10^{-32} \text{ cm}^2 \quad (40)$$

with an uncertainty of $\sim 50\%$.

At the same time, the pion production process (11) has a cross section less than $1.1 \times 10^{-32} \text{ cm}^2$ on the 90% confidence level, i.e., it does not ex-

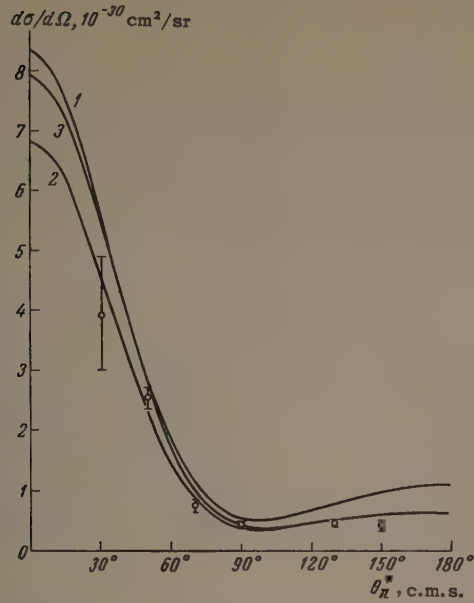


FIG. 8. Results of the calculations of the differential cross sections of the $p + d \rightarrow H^3 + \pi^+$ reaction for 340-Mev protons with allowance for the production of all positive pions in pp and pn collisions for a repulsive-core radius $r_0 = 0.6 \hbar/m_\pi c$: 1 — for a H^3 wave function of the form (41.1), 2 — for a H^3 wave function of the form (41.2), 3 — for a H^3 wave function of the form (41.3); $\circ$ — data of [26].

ceed the cross section for the electromagnetic process (12). These facts are evidence that the law of conservation of total isotopic spin is valid for processes of pion production, since if reaction (11) were not forbidden the cross section would be about 100 times the cross section of process (12).

B. Comparison with the $p + He^3 \rightarrow He^4 + \pi^+$ reaction. The upper limit found for the cross section of reaction (11) can, moreover, be compared with the cross section of reaction (13), which is allowed by the isotopic spin selection rules. In the present experiment, the cross section of reaction (13) was measured for a single l.s. angle with a proton en-

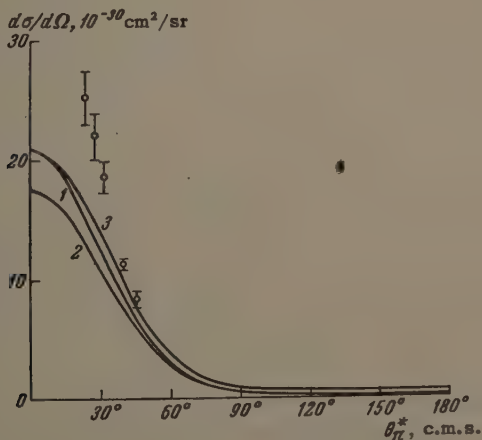


FIG. 9. Same as Fig. 8, but for 450-Mev protons; $\circ$ — data of [7].

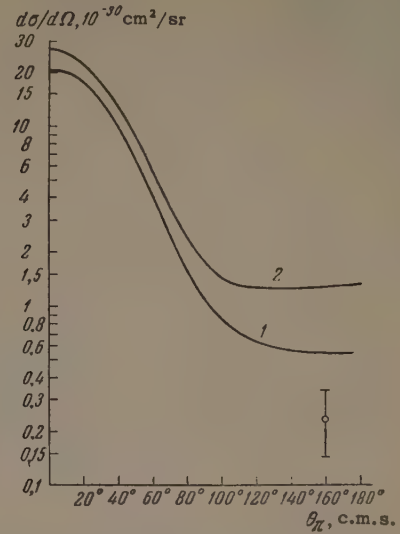


FIG. 10. Results of calculations of the differential cross section of the $p + He^3 \rightarrow He^4 + \pi^+$ reaction for 326-Mev(c.m.s.) π^+ mesons: 1 — for wave functions of He^3 and He^4 of the form (37), 2 — for wave functions of He^3 and He^4 of the form (41.3); $\circ$ — data of the present experiment.

ergy of 670 Mev, which corresponds to a π^+ c.m.s. energy of 326 Mev. In order to estimate the cross section of this reaction for a pion energy of 80 Mev, we calculated the cross section of reaction (13) in the impulse approximation. It was assumed here that the pion is produced in an elementary nucleon-nucleon collision and all the nucleons then undergo fusion into He^4 .

As a qualitative check of this method of calculation, we computed the cross section of reaction (6), which has been well studied experimentally for various proton energies.^[6-8] Such calculations were first made for 340-Mev protons.^[4] Later on,^[24] the hard-core model was used in these cal-

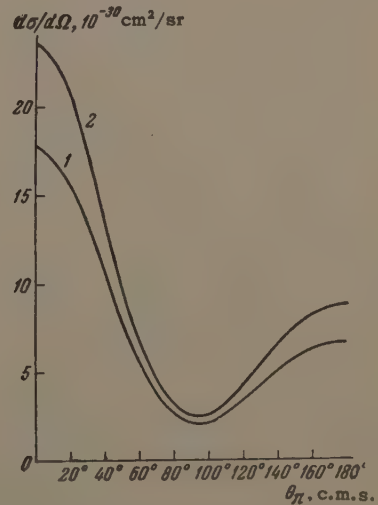


FIG. 11. Same as Fig. 10, but for π^+ mesons of energy 80 Mev in the c.m.s.

culations. In the present experiment, we calculated the cross section of reaction (6) for 340-Mev protons for the following four types of nuclear wave functions:

$$H^3 \left(R = \left(\sum_{i < j} r_{ij}^2 \right)^{1/2} \right): \quad 1) \Psi \sim \exp \left(-\frac{1}{2} \alpha \sum_{i < j} |r_{ij}| \right),$$

$$2) \Psi \sim e^{-\alpha R}, \quad 3) \Psi \sim e^{-\alpha R/R}, \quad 4) \Psi \sim e^{-\alpha^2 R^2} \quad (41)$$

with different radii of the hard core.

The increase in the radius of the repulsive core decreases the total cross section of the reaction and increases the forward-back asymmetry in the angular distribution. For a core radius greater than $0.4 \hbar/m\pi c$, the wave functions of the first three shapes give approximately the same angular dependence, in qualitative agreement with the experimental data. The Gaussian function (4) does not give agreement with the experimental data. For this reason we did not use a wave function of this shape in the further calculations.

If similar calculations are carried out for 450-Mev protons, then it appears that the angular distribution of reaction (6) is described correctly, but the total cross section differs strongly from the observed one, and this difference increases with energy. The reason for this difference is that the calculations took into account the production of π^+ mesons only in reaction (3) and did not at all take into account the production of π^+ mesons in the reactions

$$p + p \rightarrow n + p + \pi^+, \quad (42)$$

$$p + n \rightarrow n + n + \pi^+, \quad (43)$$

whose relative contribution^[25,1] increases strongly with the proton energy. The contribution of reactions (42) and (43) can be taken into account qualitatively if we add to the cross section of reaction (3) the amount by which the sum of the total cross sections for the production of π^+ mesons in reactions (42), (43), and (3) exceeds the total cross section of reaction (3). The differential cross sections of reaction (6) calculated in this way for 340- and 450-Mev protons with a repulsive-core radius of $0.6 \hbar/m\pi c$ are in satisfactory agreement with the experimental data (Figs. 8 and 9). For 340-Mev protons the wave function of the form (37) gives the best description of the experimental data.

We calculated by the same method the differential cross section of reaction (13). The calculations were made for π^+ mesons at 80 and 326 Mev in the c.m.s., which correspond to bombarding protons of 280 and 670 Mev, respectively. The results of the calculations are shown in Fig. 10 and 11. For 670-Mev protons quite good agreement was observed

between the calculated and experimental data, within the limits of a factor of 2, if a wave function of the form (37) is used. The fact that there was agreement for quite large π^+ energies gives reason to believe that the results of the calculations for 80-Mev π^+ mesons are sufficiently reliable. For this energy and for a wave function of the form (37) we obtain

$$\sigma_t(p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+) = 79 \cdot 10^{-30} \text{ cm}^2. \quad (44)$$

This result must still be compared directly with the upper limit of the cross section of reaction (11), since these two reactions differ from one another in the structure of the colliding nuclei. It is natural, however, to assume that the difference between reactions (11) and (13) is the same as that between the reactions $p + H^3 \rightarrow \text{He}^4 + \gamma$ and $d + d \rightarrow \text{He}^4 + \gamma$, i.e., the cross sections differ by a factor of 10^2 . In this case the reference point for the "allowed" reaction (11) is the cross section $0.01 \sigma_t(p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+) = 80 \times 10^{-32} \text{ cm}^2$. This value is approximately 70 times the upper limit of the total cross section of reaction (11), which is an additional indication that reaction (11) is strongly forbidden.

C. Estimate of the cross section of the "allowed" reaction $d + d \rightarrow \text{He}^4 + \pi^0$. As was indicated in the introduction, reaction (11) is forbidden only by the law of conservation of total isotopic spin, and the possible transitions in this reaction are the same as for both the forbidden process $(T = 0) \rightarrow (T = 1)$, and for the "allowed" process $(T = 0) \rightarrow (T = 0)$. By the "allowed" reaction (11) we understand a reaction in which we ascribe to the π^0 meson an isotopic spin equal to zero, while its remaining properties are unchanged. We can then carry out the calculation in the impulse approximation using a procedure similar to that of the preceding section. It was assumed in the calculations that the π^0 meson is produced in an elementary nucleon-nucleon collision and then all the nucleons undergo fusion into the He^4 nucleus. We took into account the production of π^0 mesons in both pn and pp collisions. The total cross section of the "allowed" reaction (11) calculated for a hard-core radius 0.6 and $\hbar/m\pi c$ leading to the best agreement between calculation and experiment in reaction (6) proved to be $30 \times 10^{-32} \text{ cm}^2$ for an α -particle wave function of the form (37) and $80 \times 10^{-32} \text{ cm}^2$ for an α -particle wave function of the form (41.3). The first of these estimates is the more reliable one, since it corresponds to a wave function of exponential form (37) which gives the best agreement between

calculation and experiment for reactions (6) and (13).

Hence the upper limit of the total cross section for reaction (11) is smaller by a factor of several tens than the cross section of the "allowed" reaction. The foregoing estimate for the "allowed" reaction is nothing else but an estimate of the cross section of the $d + d \rightarrow \text{He}^4 + \pi_0^0$ reaction^[10] with the emission of a π_0^0 meson with scalar isotopic spin and mass close to the mass of an ordinary π^0 meson with a vector isotopic spin. If it is assumed here that the hypothetical π_0^0 meson has the same strong-interaction constant as an ordinary pion, then the fact that reaction (11) is strongly forbidden also means that the isotopically scalar π_0^0 meson with a rest mass in the interval 100–150 Mev does not exist.

CONCLUSIONS

1) The total cross section of the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction for π^0 mesons of c.m.s. energy ~ 80 Mev does not exceed the cross section for the electromagnetic process $d + d \rightarrow \text{He}^4 + \gamma$, while the expected ratio of the cross sections of these processes should be approximately 10^2 if it were not forbidden.

2) The total cross section of the $p + \text{He}^3 \rightarrow \text{He}^4 + \pi^+$ reaction for the same pion c.m.s. energy is approximately 7×10^3 times the upper limit of the cross section for the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction, and this difference cannot be explained only in terms of the structure of the colliding nuclei.

3) The upper limit of the total cross section for the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction is $\sim 3\%$ of the calculated value of the cross section for the "allowed" process.

4) These facts taken together indicate that the $d + d \rightarrow \text{He}^4 + \pi^0$ reaction is strongly forbidden and therefore the law of conservation of total isotopic spin holds for processes of pion production by nucleons and light nuclei.

5) The π_0^0 meson with a scalar isotopic spin and rest mass of 100–150 Mev does not exist.

In conclusion, the authors express their gratitude to L. I. Lapidus for discussion of the proposed plan of the experiments, to V. P. Dzhelepov for his interest in the work and for assistance in carrying out the experiments, to R. M. Sulyaev and B. S. Neganov for aid in the work with the He^3 gas, and to Kim Ze P'he and I. V. Puzynin of the Joint Institute of Nuclear Research computer group for performing the calculations on the electronic computer.

¹A. F. Dunaitsev and Yu. D. Prokoshkin, JETP **38**, 747 (1960), Soviet Phys. JETP **11**, 540 (1960).

²R. H. Hildebrand, Phys. Rev. **89**, 1090 (1953).

³Flyagin, Dzhelepov, Kiselev, and Oganessian, JETP **35**, 854 (1958), Soviet Phys. JETP **8**, 592 (1959).

⁴M. Ruderman, Phys. Rev. **87**, 383 (1952).

⁵H. S. Köhler, Phys. Rev. **118**, 1345 (1960).

⁶Bandtel, Frank, and Moyer, Phys. Rev. **106**, 802 (1957).

⁷Creve, Ledley, Lillethan, Marcowitz, and Rey, Phys. Rev. **118**, 1091 (1960).

⁸Harting, Kluyver, Kusumegi, Rigopoulos, Sacks, Tibell, Vanderhaeghe, and Weber, Phys. Rev. **119**, 1716 (1960).

⁹L. I. Lapidus, JETP **31**, 865, (1956), Soviet Phys. JETP **4**, 740 (1957).

¹⁰A. Baldin and P. Kabir, DAN SSSR **122**, 361 (1958), Soviet Phys.-Doklady **3**, 956 (1958).

¹¹Akimov, Savchenko, and Soroko, JETP **38**, 304 (1960), Soviet Phys. JETP **11**, 221 (1960).

¹²Akimov, Savchenko, and Soroko, JETP **35**, 89 (1958), Soviet Phys. JETP **8**, 64 (1959).

¹³B. S. Neganov and L. B. Parfenov, JETP **34**, 767 (1958), Soviet Phys. JETP **7**, 528 (1958).

¹⁴B. H. Flowers and F. Mandl, Proc. Roy. Soc. (London) **206**, 131 (1951).

¹⁵M. G. Meshcheryakov and B. S. Neganov, DAN SSSR **100**, 677 (1955).

¹⁶I. M. Vasilevskii and Yu. D. Prokoshkin, Atomnaya Énergiya (Atomic Energy) **7**, 225 (1959).

¹⁷Akimov, Savchenko, and Soroko, Pribery i Tekhnika Éksperimenta (Instrum. and Exptl. Techniques) No. 4, 71 (1960).

¹⁸Akimov, Savchenko, and Soroko, JETP **38**, 643 (1960), Soviet Phys. JETP **11**, 462 (1960).

¹⁹A. N. Gorbunov and V. M. Spiridonov, JETP **33**, 21 (1957), Soviet Phys. JETP **6**, 16 (1958).

²⁰Keck, Tollestrup, and Smythe, Phys. Rev. **96**, 850 (1954).

²¹J. Irving, Phil. Mag. **42**, 338 (1951).

²²A. V. Tollestrup, Phys. Rev. **75**, 1947 (1949).

²³R. W. McAllister and R. Hofstadter, Phys. Rev. **102**, 851 (1956).

²⁴S. Bludman, Phys. Rev. **94**, 1722 (1954).

²⁵B. S. Neganov and O. V. Savchenko, JETP **32**, 1265 (1957), Soviet Phys. JETP **5**, 1033 (1957).

²⁶Frank, Bandtel, Medey, and Moyer, Phys. Rev. **94**, 1716 (1954).

Translated by E. Marquit

128

NEUTRON POLARIZATION IN THE REACTION $T(d, n)He^4$

I. S. TROSTIN, V. A. SMOTRYAEV, and I. I. LEVINTOV

Institute of Theoretical and Experimental Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 22, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 725-727 (September, 1961)

The azimuthal asymmetry in the scattering on He nuclei of neutrons from the reaction $T(d, n)He^4$ ($E_d = 9.9 \pm 0.7$ Mev) is measured for various angles of neutron emission θ_n . A marked asymmetry is found at $\theta_n = 70^\circ$ (laboratory system). The azimuthal asymmetry is measured at this emission angle as a function of the angle of scattering on He^4 . The results are compared with $n-He^4$ (Seagrave) and $p-He^4$ (Gammel-Thaler) scattering phase shifts. The polarization of the investigated neutrons estimated on the basis of the Gammel-Thaler phase shifts is $P_T(70^\circ) = (+32.1 \pm 3.0)\%$ (the direction along the normal $\mathbf{k}_n \times \mathbf{k}_d$ is considered positive).

THE weak polarization of neutrons from the $T(d, n)He^4$ reaction (deuteron energy $E_d = 1.8$ Mev) has been previously^[1] ascertained. The present paper investigates the possibility of using the same reaction as a source of polarized neutrons for $E_d \approx 10$ Mev.

A beam of 12.3-Mev deuterons from the Institute of Theoretical and Experimental Physics cyclotron was focused on a zirconium target 0.6 Mev thick which was saturated with tritium. The average current was 1.5 μ amp with a 5×3 mm spot of the beam on the target. A 103-mg/cm² platinum foil was laid directly against the target to separate it from the working volume of the cyclotron.

The azimuthal asymmetry of the neutron scattering was measured in a helium analyzer described earlier.^[2,3] The helium pressure in the proportional counters was varied from 7 to 20 atm, depending on the neutron emission angle, and was held constant within $\pm 0.5\%$. The length of the effective volume of the counters was ~ 20 mm, and the working voltage was 600–1200 v. To maintain the gas amplification factor constant, a continuous convective flow of helium was maintained in a metallic-calcium column, connected to the counters and heated to 300°.

Departing from previous practice,^[1-3] the proportional counters were calibrated with Po^{210} α particles. The polonium was deposited on a platinum foil, which was fitted tightly to the inner walls of the counter. The presence of a constant background of Po^{210} α particles had no effect on the asymmetry measurements, since their energy was less than the energy of the recoil He^4 nuclei over the whole of the angular interval used. A fission

chamber was used as a monitor, and a current integrator applied to measure current passing through the target.

We did not know in advance either the angles at which the neutrons were polarized in the $T(d, n)He^4$ reaction or the $n-He^4$ scattering phase shifts at neutron energies $E_n \gtrsim 20$ Mev. Preliminary measurements of the azimuthal scattering asymmetry were therefore made as a function of the neutron emission angle (θ_n) with the counters set at $\varphi_\alpha = \pm 35^\circ$ [$\varphi_\alpha = (\pi - \varphi_n)/2$, where φ_n is the scattering angle of the neutrons on He^4 nuclei in the center-of-mass system (cf. Fig. 1 in ^[2])]. This angle was chosen from polarization curves for $p-He^4$ scattering at $E_p \gtrsim 18$ Mev.^[4] As a result, the following values for the azimuthal asymmetry $R = I_1/I_2$ are obtained (I_1 and I_2 are the counting rates in the directions $+\varphi_\alpha$ and $-\varphi_\alpha$, the + sign referring to the case when φ_α is taken in the same direction as θ_n):

θ_n (l.s.), deg:	30	50	60	70	90
E_n , Mev:	25.8	23.6	22.2	20.7	17.9
R:	1.03 ± 0.03	1.12 ± 0.03	1.06 ± 0.04	1.34 ± 0.09	1.06 ± 0.08

Since the measurements were made in a "good geometry",^[3] anisotropy of the angular distribution in the $T(d, n)He^4$ reaction was not taken into account (the correction was $\sim 1\%$). The target construction permitted measurement of the background associated with neutrons coming from the diaphragms, the target backing, etc. The background was negligibly small at small θ_n ; at $\theta_n = 90^\circ$ (lab. system) it was $\sim 15\%$ of the counts in the utilized analyzer channel.

The second stage of the work consisted of measuring the dependence of the azimuthal asym-

metry on the angle φ_n of neutron scattering on He^4 nuclei at the angle $\theta_n = 70^\circ$ (lab. system) of neutron emission from the target. R and the product of the polarizations for the given reaction and analyzer are connected by the well-known relation:

$$P_T(\theta_n) P_{He}(\varphi_n) = (1 - R) / (1 + R),$$

where $P_T(\theta_n)$ is the neutron polarization in the

φ_n (l.s.), deg	90	110	124	136	150
R	0.71 ± 0.05	1.34 ± 0.09	1.53 ± 0.15	1.50 ± 0.10	1.26 ± 0.15
$-P_T(70^\circ) P_{He}(\varphi_n)$, %	-17.0 ± 3.4	14.5 ± 3.3	20.9 ± 4.5	20.0 ± 3.2	11.5 ± 5.9

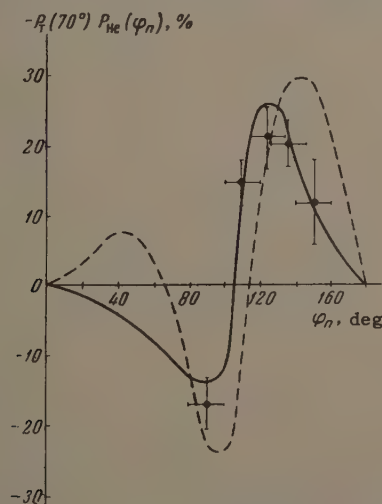
Only statistical errors are indicated here.

The figure shows values obtained for $P_T(70^\circ) \times P_{He}(\varphi_n)$ as well as the results of calculating the same quantity from phase shifts for $n-He^4$ scattering (according to Seagrave^[5]) and $p-He^4$ scattering (according to Gammel-Thaler^[6]). The spread of angles of the recoil He^4 nuclei inside the counters was taken into account in the calculation of the curves.

The best agreement between calculated curves and experimental results is attained when the polarization $P_{He}(\varphi_n)$ is computed from Gammel-Thaler phase shifts^[6] and the neutron polarization in the reaction $T(d, n)He^4$ has the value

$$P_T(70^\circ) = (32.1 \pm 3.0) \%$$

The figure indicates a definite disagreement between our results and the Seagrave phases at $E_n \approx 20$ Mev. It must, however, be observed that the Seagrave analysis does not claim to give a quantitative agreement at such energies. On the other hand, the agreement of our $n-He^4$ scattering data with calculations from $p-He^4$ scattering phase shifts (which are based on more significant experi-



$-P_T(70^\circ) P_{He}(\varphi_n)$ as a function of the angle of neutron scattering on He^4 ($E_n = 20.7 \pm 0.4$ Mev). The solid curve is calculated from Gammel-Thaler phase shifts,^[6] the dashed from Seagrave phase shifts.^[5] The curves are normalized to the polarization $P_T(70^\circ) = 32.1\%$.

reaction $T(d, n)He^4$ (neutrons emitted at an angle of θ_n to the target); $P_{He}(\varphi_n)$ is the polarization of neutrons scattered on He^4 at an angle of φ_n ; and R is the azimuthal asymmetry in the scattering. The polarization is considered positive in the direction $\mathbf{n} = \mathbf{k}_n \times \mathbf{k}_d$.

The following results were obtained:

mental material) is not strange, since in large momentum transfers the angular dependences of polarization in $p-He^4$ and $n-He^4$ scattering should not be very different.

Thus our results may be considered as an experimental confirmation of Gammel-Thaler phase shifts at $E_n \sim 20$ Mev. If this is so, the small azimuthal asymmetry in $n-He^4$ scattering observed at $\varphi_n = 35^\circ$ for neutrons coming at angles $\theta_n < 70^\circ$ from the $T(d, n)He^4$ reaction is actually evidence of small polarization at these angles.

There is the hypothesis^[7] that the reaction $T(d, n)He^4$ proceeds by the stripping mechanism at small angles θ_n . In this case the polarization in the region of angles around the Butler peak must be small for this reaction, since the angular momentum of the captured proton $l_p = 0$. Neutron polarization at large emission angles can arise as a result of a spin-orbit interaction between departing neutron and α particle.^[8,9]

The authors express their thanks to the cyclotron crew of the Institute of Theoretical and Experimental Physics for the efficient operation of the accelerator, and to V. S. Repin and E. A. Mikhaleva for their aid in the measurements.

¹ Levintov, Miller, and Shamshev, JETP **34**, 1030 (1958), Soviet Phys. JETP **7**, 712 (1958).

² Levintov, Miller, and Shamshev, JETP **32**, 274 (1957), Soviet Phys. JETP **5**, 258 (1957).

³ I. I. Levintov and I. S. Trostin, JETP **40**, 1570 (1961), Soviet Phys. JETP **13**, 1102 (1961).

⁴ K. Brockman, Phys. Rev. **102**, 391 (1956).

⁵ J. Seagrave, Phys. Rev. **92**, 1221 (1953).

⁶ J. L. Gammel and R. M. Thaler, Phys. Rev. **109**, 2041 (1958).

⁷ F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959).

⁸ H. C. Newns and M. Y. Refai, Proc. Phys. Soc. **71**, 627 (1958).

⁹ Bokhari, Cookson, Hird, and Weesakul, Proc. Phys. Soc. **72**, 88 (1958).

SUPERCONDUCTING PROPERTIES OF FRESHLY DEPOSITED MERCURY FILMS

I. S. KHUKHAREVA

Moscow State University

Submitted to JETP editor April 27, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 728-734 (September, 1961)

Mercury films obtained by the method of condensation in high vacuum on a backing at a temperature close to that of liquid helium have been studied. The experimental results are compared with the Abrikosov-Gor'kov theory of superconducting alloys.

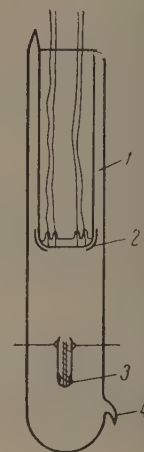
THE study of freshly deposited metal films, condensed at liquid-helium temperature,^[1-5] is of considerable interest in that it appears that a material is obtained which does not possess very sharply pronounced peculiarities, although it is very different from the initial material. From the very high resistivity and the absence of a dependence of conductivity on thickness^[3,5] in freshly deposited films in the normal state, it can be assumed that in the condensation process a deposit is formed with isotropic structure, in which the electron mean free path is considerably less than the thickness of even the thinnest films.

Mercury is most convenient as a substance to study for establishing the process of low temperature condensation, since, with a relatively high vapour pressure, sufficiently thick specimens can be obtained easily while using a very small power in the evaporation. Appleyard et al.^[6] have previously studied the specific resistance and the superconducting properties of mercury films, condensed on a backing at a temperature from 4.2 to 90° K. However, films condensed at 4.2° K have not been studied systematically.

PREPARATION OF THE SPECIMENS AND THE MEASUREMENTS

The mercury films were prepared in the apparatus shown in Fig. 1. The backing for the condensation was the outer plane polished base of a glass beaker, through which platinum leads are inserted. Before being sealed into the apparatus, the beaker was thoroughly treated with a mixture of nitric and chromic acids and boiled in distilled water. A monel shield was fitted to the beaker, leaving open only the middle part of the base of the beaker for condensation. A thin walled glass capsule ~ 10 mm in length with an exit opening 1.0 - 1.5 mm in diameter served as the evaporator, and the platinum wire was wound on the outside and half

FIG. 1. Apparatus for obtaining the films: 1 - beaker with plane polished base, 2 - monel shield, 3 - capsule with mercury, 4 - side tube for filling the apparatus with liquid helium.



sealed into the glass for better thermal contact. The power generated in the evaporator did not exceed 0.2 watt. A thin glass capillary was fixed into the capsule to avoid bubbles, which could form during the working of the evaporator. The distance between the orifice of the evaporator and the bottom of the beaker was varied from 30 to 50 mm in different apparatus. The assembled apparatus was pumped by a diffusion pump to 10^{-6} mm Hg, and its upper half was baked in an oven for 3-4 hours at a temperature of 300° C.

Condensation of the films took place at 4.2° K (the boiling point of helium) at a rate of $(1.1 - 1.3) \times 10^{-3}$ g/min. The thickness of the films obtained was calculated from the mass of mercury evaporated, on the assumption that the distribution of current density of the particles evaporating follows the cosine law. As a check on the calculated thickness, an experiment was made in which one of the thickest films was collected from the bottom of the beaker, after the heating, and weighed. The thickness, determined in this experiment, was 20% larger than calculated. This discrepancy can evidently be explained by the existence of a directed particle current, not taken into account by the cosine law.

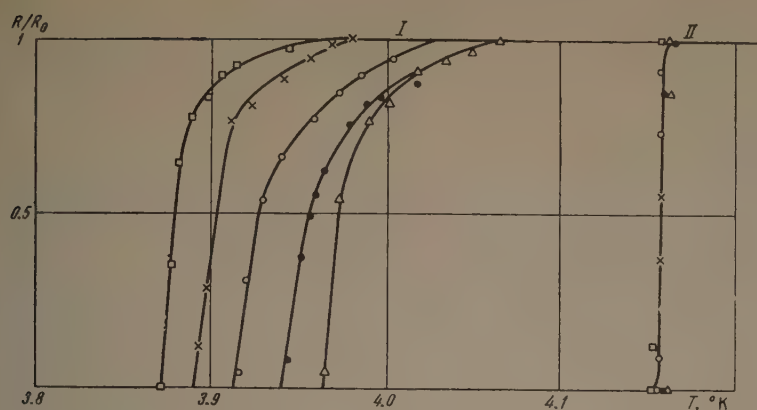


FIG. 2. The temperature dependence of the resistance of the films: I – freshly deposited films, II – annealed films. Film thicknesses in 10^{-5} cm: Δ – 0.46, $\bullet$ – 0.58, $\circ$ – 0.89 $\times$ – 2.8, $\square$ – 4.0.

Films with thicknesses from 0.46×10^{-5} to 12.2×10^{-5} cm (these thicknesses are calculated on the assumption that the cosine law applies) were produced and studied. Measurements were carried out both immediately after condensation of the films and after they had been heated to liquid-nitrogen temperature. The resistance of the films was measured with a standard potentiometer circuit, using a measuring current not exceeding $100 \mu\text{a}$. Temperature was determined from the helium saturated-vapor pressure. The temperature dependence of resistivity in zero magnetic field was measured both immediately after condensation and on cooling the films down a second time. These curves for films of different thicknesses are shown in Fig. 2.

The transition temperatures of the freshly deposited films lie somewhat lower than that for bulk mercury (we took the temperature in the

middle of the transition as the transition temperature of a film; $R = 0.5 R_0$, where R_0 is the resistance in the normal state). The observed dependence of transition temperature on thickness agrees well with Prozorova's results.^[5] The transition temperature for the annealed films practically coincides with the value of the transition temperature for bulk mercury (4.15°K) and is independent of film thickness (see Fig. 3); the superconducting transitions are sharper than for freshly deposited films.

Other physical properties of the films naturally change after annealing also. For example, the resistivity in the normal state decreases more than 10 times. The form of the variation of resistivity with film thickness also changes. It remains constant in freshly deposited films up to the smallest of the thicknesses studied. The conductivity of the annealed films, however, increases proportionally with their thickness. The absolute

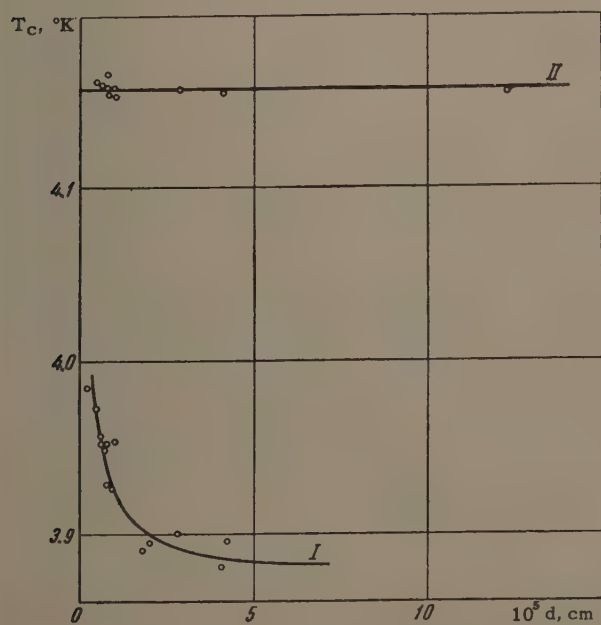


FIG. 3. The dependence of the transition temperature of a film on its thickness: I – freshly deposited films, II – annealed films.

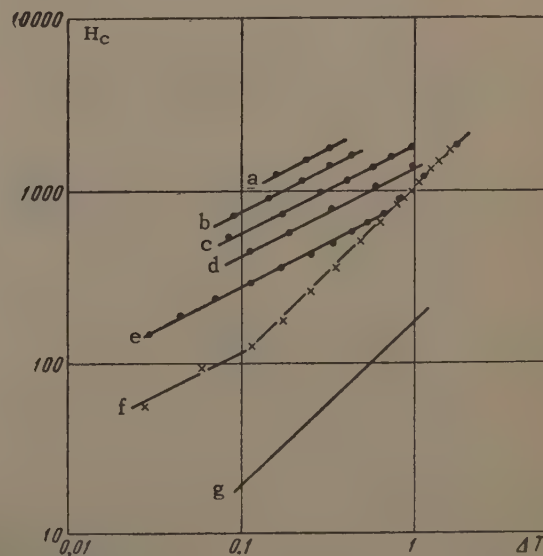


FIG. 4. The dependence of H_c on ΔT ($\Delta T = T_c - T$) for films of different thickness (in 10^{-5} cm): a – 0.47, b – 0.58, c – 0.76, d – 0.90, e – 2.8, f – 4.2, g – bulk specimen.

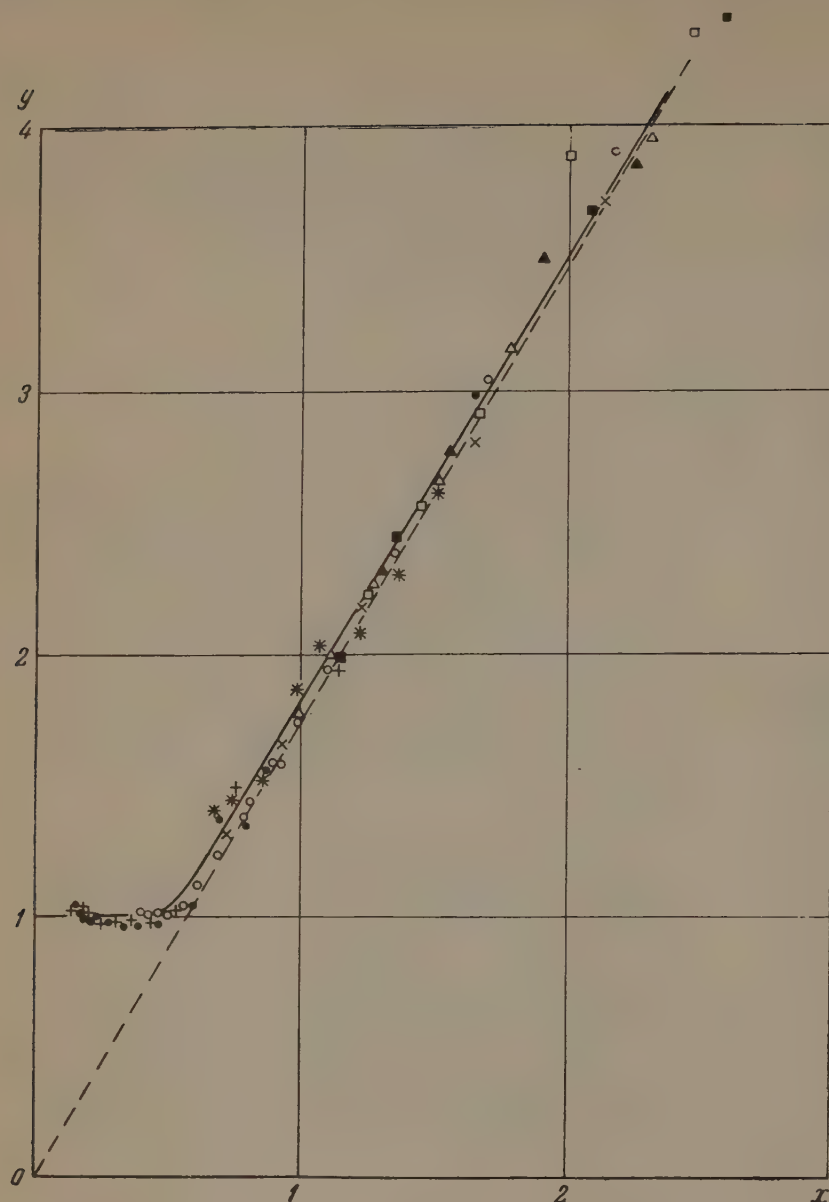


FIG. 5. The dependence of $y = H_c/2\theta^2$ on $x = 2.57 \times 10^{-4}/\theta d$ for films of thickness (in 10^{-5} cm): + - 4.2, $\bullet$ - 4.0, $\circ$ - 2.8, * - 1.0, $\times$ - 0.9, $\blacksquare$ - 0.83, $\square$ - 0.81, $\triangle$ - 0.76, $\blacktriangle$ - 0.72. Full line is the theoretical curve.

value of the conductivity of a freshly deposited film, measured in a separate experiment, was $\sigma = 0.15 \times 10^{18}$ cgs esu.

Curves of the destruction of superconductivity by a magnetic field, parallel to the plane of the films, were also measured for all the films obtained. Accuracy in setting the film parallel to the field was achieved by rotating the field, and did not exceed 0.1° . Figure 4 shows curves of $H_c(\Delta T)$ ($\Delta T = T_c - T$) for freshly deposited films of different thickness. The field at which the film resistance reached $0.5R_0$ was taken as H_c . As is seen from Fig. 4, $H_c \sim T^{1/2}$ for thin specimens, while a break is observed in the curves of the dependence of H_c on ΔT for thick films.

DISCUSSION OF THE RESULTS

The macroscopic theory of Ginzburg and Landau [7] was developed by Abrikosov [8] in a way

applicable to the case of films obtained by low temperature condensation, on the assumption that the condition $\kappa \geq 1/\sqrt{2}$ (κ is the parameter of the Ginzburg-Landau theory) is valid for them. Such an assumption seems quite justifiable, since films condensed on a cooled backing show properties characteristic of alloys. Abrikosov obtained a universal relation, for all superconductors, between the critical magnetic field H_c for a film and its thickness d . Taking into account that $\kappa = (\sqrt{2}e_{\text{eff}}/\hbar c)H_{cb}\delta_0^2$ where $e_{\text{eff}} = 2e$, this relation can be written

$$H_c/2\theta^2 = f(2.57 \cdot 10^{-4}/\theta d),$$

where $\theta = \sqrt{2e/\hbar c}H_{cb}\delta_0$ and H_{cb} is the critical field of bulk metal.

We assume that near T_c

$$H_{cb} = (dH_{cb}/dT)_c \Delta T; \quad \delta_0 = \delta_{00} \sqrt{T_c}(\Delta T)^{-1/2}/2.$$

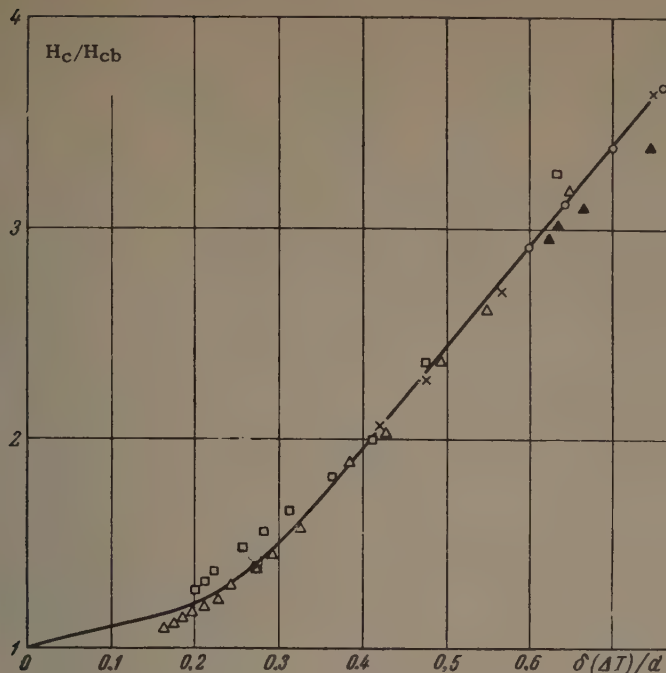


FIG. 6. The ratio H_c/H_{cb} as a function of $\delta(\Delta T)/d$. Film thickness in 10^{-5} cm: Δ — 4.9, $\square$ — 4.1, $\times$ — 2.8, $\circ$ — 1.0, $\blacktriangle$ — 0.9. The continuous curve shows the theoretical relation.

Then

$$\theta = A\Delta T^{1/2}, \quad A = \sqrt{e/2\hbar c} (dH_{cb}/dT)_{T_c} \sqrt{T_c} \delta_{00}.$$

It can be seen from Fig. 5 that the experimental results are in good agreement with the above theory.

The constant, A , can be determined from the part of the curve corresponding to the limiting case $d/\delta_0 \gg 1$, and the constant coefficient δ_{00} in the temperature dependence of the penetration depth near T_c can be calculated. From our data $\delta_{00} = (22.6 \pm 2.3) \times 10^{-6}$ cm.

In the other limiting case, when $d/\delta_0 \ll 1$, we have

$$H_c/2\theta^2 = \sqrt{3} \cdot 2.57 \cdot 10^{-4} / \theta d.$$

From this, the film thickness d can be determined. The calculation made for the thickest film of all agrees, within the limits of accuracy, with the value of the thickness obtained as a result of weighing the film itself directly.

From the new Abrikosov-Gor'kov theory of superconducting alloys^[9] the parameter δ_{00} can be calculated from the value of the normal conductivity. It follows from this theory that for $1 \ll \delta_0$ (where l is the mean free path and δ_0 the penetration depth)

$$\delta_0(\Delta T) = \frac{c}{3.06\pi} \sqrt{\frac{\hbar}{2k\sigma}} \Delta T^{-1/2}, \quad \delta_{00} = \frac{c}{3.06\pi} \sqrt{\frac{2\hbar}{kT_c\sigma}},$$

where σ is the conductivity in the normal state. We obtained for freshly deposited films the values

$$\sigma = 0.15 \cdot 10^{18} \text{ cgs esu}, \quad \delta_{00} = 16 \cdot 10^{-6} \text{ cm}.$$

This value of δ_{00} is somewhat lower than that obtained from data on critical fields. However, this discrepancy can be explained if we take into account that the calculation of σ was made for a film which was close to the evaporator when being produced, and in which, therefore, there might have been partial recrystallization and, consequently, an increase in the value of σ .

The experimental data on the critical fields of the annealed films agree well with deductions from the Ginzburg-Landau macroscopic theory.^[7] The dependence of the critical fields of the films on thickness follows the following two equations in the limiting cases $d/\delta_0 \gg 1$ and $d/\delta_0 \ll 1$ respectively

$$H_c/H_{cb} = 1 + \delta_0/d, \quad H_c/H_{cb} = 2\sqrt{6} \delta_0/d.$$

The results of the confirmation of these relations are shown in Fig. 6.

Considering the region $T \rightarrow T_c$, where we can write

$$H_{cb} = (dH_{cb}/dT)_{T_c} \Delta T, \quad \delta_0 = \sqrt{T_c} \delta_{00} \Delta T^{-1/2}/2,$$

we calculated the value of the parameter from the experimental values of H_c for thin films ($d \ll \sqrt{5} \delta_0$),

$$\delta_{00} = (6.2 \pm 0.6) \cdot 10^{-6} \text{ cm}.$$

In conclusion I thank A. I. Shal'nikov, under whose direction this work was carried out, very sincerely, and A. A. Abrikosov for discussion of the results obtained.

¹A. I. Shal'nikov, JETP 10, 630 (1940).

²H. E. Alekseevskii, JETP 10, 1392 (1940)

³W. Buckel and R. Hilsch, Z. Physik 138, 109 (1954).

⁴N. V. Zavaritskii, Doklady Akad. Nauk SSSR 82, 229 (1952) and 86, 501 (1952).

⁵L. A. Prozorova, JETP 34, 14 (1958), Soviet Phys. JETP 7, 9 (1958).

⁶Appleyard, Bristow, London, and Misener, Proc. Roy. Soc. A172, 540 (1939).

⁷V. L. Ginzburg and L. D. Landau, JETP 20, 1064 (1950).

⁸A. A. Abrikosov, Doklady Akad. Nauk SSSR 86, 489 (1952).

⁹A. A. Abrikosov and L. P. Gor'kov, JETP 35, 1558 (1958), and 36, 319 (1959), Soviet Phys. JETP 8, 1090 (1959) and 9, 220 (1959).

ENERGY SPECTRUM AND TIME DEPENDENCE OF THE INTENSITY OF SOLAR COSMIC-RAY PROTONS

A. N. CHARAKHCH'YAN, V. F. TULINOV, and T. N. CHARAKHCH'YAN

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.;
Institute of Nuclear Physics, Moscow State University

Submitted to JETP editor April 27, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 735-746 (September, 1961)

Data obtained in the stratosphere are presented on the energy spectrum and time dependence of the total intensity of solar flare protons.

INTRODUCTION

It has been established that cosmic ray bursts recorded in northern latitudes of the stratosphere are produced by primary protons generated during large solar flares. For example, in July, 1959 three large class 3+ flares occurred separated by one and two days, respectively. Large increases of cosmic ray intensity with the same time sequence were recorded in the stratosphere during July.^[1,2] However, there are instances when it is difficult to associate a definite solar flare uniquely with a registered cosmic ray outburst. This apparently results from the fact that occasionally some relatively small flares are cosmic ray sources. Solar flares below classes 2 and 3 in importance occur more frequently, so that it is not always possible to associate individual solar flares and cosmic ray bursts uniquely. Nevertheless, cosmic ray bursts in the stratosphere that are not associated with any specified flares are also attributed to solar-emitted protons.

The pattern of effects associated with a cosmic ray burst can be outlined as follows. In most instances cosmic ray bursts are observed a few hours later than a solar flare.^[1] A period of magnetic storms, ionosphere disturbances, and in some instances aurorae begins about a day after a flare. At the onset of a magnetic storm or a little later a decrease of high-energy cosmic ray intensity is registered on the ground (the Forbush decrease). In some instances the cutoff energy of primary particles also changes.^[3] There are also cases when a cosmic ray burst in the stratosphere is not accompanied by geophysical effects. The associated flares ordinarily erupt at the edge of the solar disk, emitting corpuscular streams that miss the earth. It is naturally of interest to inves-

tigate cosmic ray intensity increases in these instances.

It has been found from the study of bursts in the stratosphere that the Forbush decrease becomes more pronounced with increasing amplitude of the bursts induced by solar protons,^[1] thus indicating that some relation exists between these two diverse events. It has also been found that sudden magnetic storms accompanying a Forbush decrease (according to measurements at sea level, where most of the cosmic ray intensity is due to high-energy primary particles of galactic origin) were not associated with reduced proton intensity in bursts.^[4] A proposed explanation is that solar protons registered during the Forbush decrease do not move outside of the corpuscular streams but are carried by the latter.^[4] Another step forward in accounting for this interesting property of solar corpuscular streams has been an investigation of the flare proton energy spectrum before and during the Forbush decrease.

If it should be assumed that during a Forbush decrease solar protons strike the earth outside of the corpuscular streams, we would expect a flatter proton energy spectrum during this time than before the Forbush decrease, as occurs in the energy spectrum of galactic cosmic rays.^[5,6] The bursts actually present the reverse of this picture, —a steep spectrum during the Forbush decrease, followed by a flat portion. This effect, which we first noticed in July, 1959, also suggested the aforementioned property of solar corpuscular streams.^[1] We shall here present new data on the energy spectrum of solar flare protons.

We also investigated the time dependence of primary proton intensity. The burst of May 4, 1960 in the stratosphere furnished data for the initial, as well as for the following, period of burst de-

Table I

Date	Time*	m	τ , hrs	Date	Time*	m	τ , hrs
April 1, 1960	10 ^h 30'	4	2	May 12, 1960	08 ^h 30'	3	2
April 28, 1960	08 30	7	12	May 13, 1960	08 30	5	6
	10 43	3			11 30	2	
	13 27	7			14 30	1	
	20 45	1		September 3, 1960	08 30	50	
May 4, 1960	11 28	34	30		13 15	70	}
	13 55	30			18 48	20	
	16 30	16		September 4, 1960	08 15	20	
	19 29	10			13 26	11	
May 5, 1960	02 30	4	30	September 5, 1960	08 23	2	}
	08 31	2.8			13 47	1.5	
	11 32	2			19 30	0.5	
	14 30	2					
	18 29	1.5					

*Universal time.

velopment. These data suggest certain conclusions regarding the propagation of low-energy solar protons in interplanetary space.

MEASUREMENTS

Measurements were obtained with a radio-sonde^[7] carried into the stratosphere by balloons. Altitude dependences were measured for single-counter readings and for double coincidences in a telescope consisting of two Geiger counters separated by 7 mm Al.

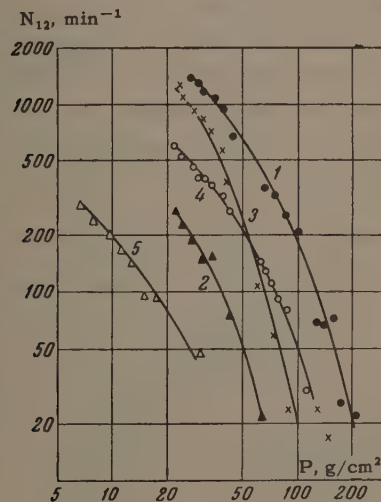


FIG. 1. Number N_{12} of double coincidences (with normal background subtracted) vs pressure P .

Curve	Date	Onset
1	September 3, 1960	07 ^h 00'
2	September 4	11 56
3	May 4, 1960	9 58
4	May 4	15 00
5	May 5	10 20

Table I contains the data on the registered bursts. The third column gives the relative intensity increase of primary radiation $m = (N_{\text{exc}} - N_0)/N_0$, where $(N_{\text{exc}} - N_0)$ is the excess, and N_0 is the normal primary cosmic ray intensity. The values of $N_{\text{exc}} - N_0$ were obtained by extrapolating the measurements to a pressure of 5 g/cm². The fourth column gives the duration τ of the observed bursts. The events of May 4, 12, and 13, 1960 were not accompanied by a Forbush decrease.

The event of April 1 was registered by the Winckler^[8] and Simpson^[9] groups, with equipment installed in the American space probe Pioneer V, at 5×10^8 km from the earth, and by the Van Allen group^[10] using the American satellite Explorer VII. The combined data for the event of April 1 provide good confirmation of the extraterrestrial origin of the stratosphere bursts. The event of May 4-5, 1960 was also registered on the ground.

SPECTRUM OF PRIMARY PROTON RANGES

The bursts of May 4 and September 3, 1960 are of greatest interest with regard to the energy spectrum of primary protons and the time dependence of their intensity. These bursts had large amplitudes and the most prolonged duration.

Figures 1 and 2 show the measurements obtained with a telescope and with a single counter during the events of May 4 and September 3. The experimental points are the averages over three minutes. The slopes of the absorption curves for different times during a burst differ only slightly. The steeper slope of curve 3 is associated with the time dependence for the beginning of the event (see below). Curve 2 in Fig. 1 was obtained 29 hrs after the measurements on September 3 (curve 1). After this period the amplitude of the burst was

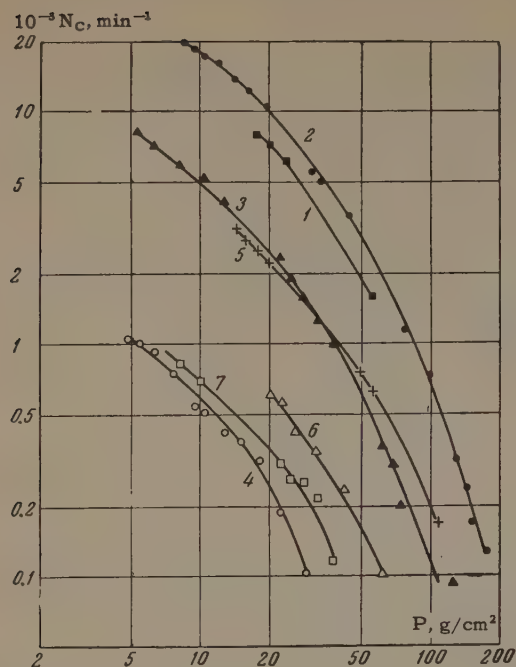


FIG. 2. Proton count N_C (after background subtraction) obtained with a single counter vs pressure P .

Curve	Date	Onset
1	September 3, 1960	07 ^h 00'
2	September 3	11 45
3	September 4	6 45
4	September 5	6 53
5	May 4, 1960	15 00
6	May 5	1 00
7	May 5	7 00

reduced to less than one-fourth without much change in the slope of the absorption curve. The same applies to curves 4 and 5 in Fig. 1 for May 4 and 5. Curves 1–4 in Fig. 2 for the September burst, based on measurements with a single counter, also have similar slopes.

Curves 5, 6, and 7 in Fig. 2 present a similar picture for the single-counter measurements of May 4 and 5, although segments of the curves in the long-range region exhibit a tendency toward a gradual increase of steepness. This indicates that high-energy flare protons disappear somewhat sooner than protons with lower energies. We thus arrive at the general conclusion that there is no essential change in the primary-proton absorption curve throughout the duration of a burst (30–50 hrs).

Figure 3 shows the energy spectra of primary protons in different bursts measured with the telescope. Here we took into account the ionization losses and proton absorption due to nuclear colli-

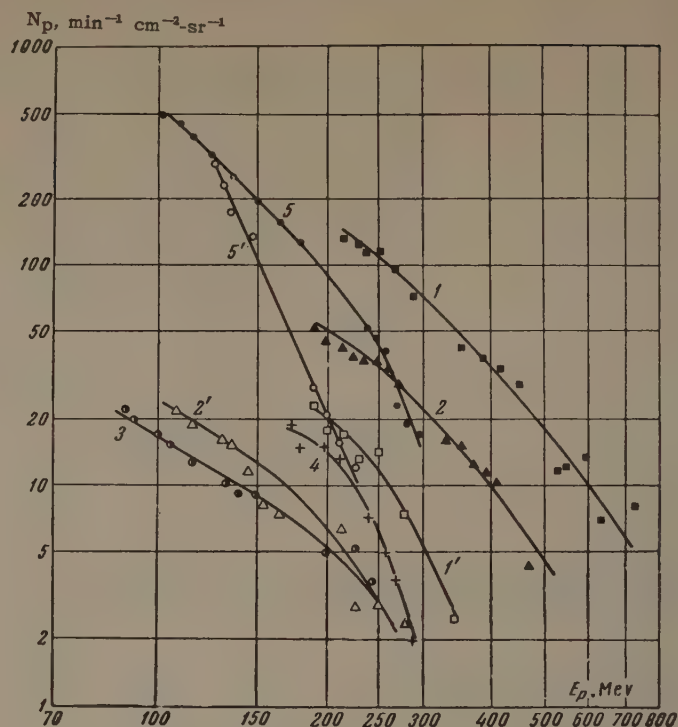


FIG. 3. Integral proton spectra. Curves 1 and 1'—September 3 and 4, 1960; 2 and 2'—May 4 and 5, 1960; 3—May 13, 1960; 4—April 28, 1960; 5 and 5'—July 11 and 12, 1959. N_P is the number of protons and E_P is the proton kinetic energy.

sions in air. The figure also includes the measurements of July 11 and 12, 1959 (curves 5 and 5'). The slope of curve 5 is somewhat less steep (for $E_P < 200$ Mev) than that shown in [1]; this is associated with the subsequent improvement of the altitude graph of the measurements.

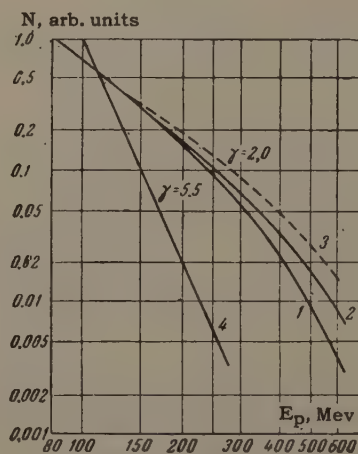


FIG. 4. Integral proton energy spectra. 1—average spectrum in the absence of magnetic storms and Forbush decrease; 2—the same, allowing for proton absorption resulting from nuclear collisions in air, as well as ionization losses; 3—derived from curve 2 by taking into account approximately the proton diffusion time in space as a function of velocity; 4—average spectrum during magnetic storms and Forbush decrease.

Table II

Date	Time	P, g/cm ²	N', min ⁻¹	N'', min ⁻¹	N'/N''	Remarks
May 4, 1960	17 ^h 03'	18	2 600	2 900	1,9	Magnetic storms and Forbush decrease absent
	16 28	53	680	580	1,2	
September 3, 1960	8 32	20	7 500	8 000	0.94	"
	8 04	57	1 600	1 600	1.0	
May 12, 1959	13 23	23	900	320	2.8	Magnetic storm and Forbush decrease present
July 12, 1959	13 42	17	2 950	1 440	2.0	"
	13 11	52	82	48	1,7	
July 15, 1959	13 20	21	13 200	2 200	6	"
	13 01	57	2 180	60	36	

The spectra in Fig. 3 (with the exception of curve 5') were registered at a time when no magnetic storm nor Forbush decrease was observed. The average spectra from these data are represented by curves 1–3 in Fig. 4. Curve 1 takes only ionization losses by protons into account. Curve 2 was derived from curve 1 by also taking into account proton absorption in nuclear collisions. The nuclear absorption coefficients for protons as functions of energy were obtained from the literature.^[11–13] The absorption coefficient in air, converted from data in nuclear emulsion and copper, is 300 g/cm² for 350–400 Mev protons^[11,12] and 170 g/cm² for 660-Mev protons.^[13] Curve 3 was derived from curve 2 by taking into account approximately the proton diffusion time in space as a function of velocity. With these corrections, the spectrum 3 in the 100–400 Mev range has an exponent γ close to 2.0. Spectrum 4, recorded during Forbush decreases (May 12, July 12, and July 15, 1959) has $\gamma \approx 5.0$.

INTENSITIES FROM TELESCOPE AND SINGLE-COUNTER MEASUREMENTS

Our measurements were performed with a single STS-6 counter having 0.05 g/cm² wall thickness, and with a telescope incorporating a 7-mm Al absorber. As reported in^[4], the emission intensity registered by the single counter considerably exceeded that registered with the telescope. This indicates that during the event particles with ranges under 7 mm Al were also present in the stratosphere. It was noted in^[4] that this effect could be attributed qualitatively to hard gamma rays from inelastic collisions of evaporation neutrons in the stratosphere. R. R. Brown of the University of California has informed us in a private communication that the unusually high intensity of 2–3 Mev photons registered by him in the stratosphere during the burst of July

15, 1959 can hardly be attributed to evaporation neutrons. Our data for July 15 also exclude this interpretation.

A Gross transformation of the number N_{12} of double coincidences gave the total proton flux at different altitudes. N_{12} and the single-counter readings were obtained during the same flight. In the absence of a Forbush decrease the single-counter and telescope measurements agreed. However, during a Forbush decrease the single-counter intensities exceeded the expected transformed values. This is illustrated by Table II, where P is the pressure, N' is the single-counter reading, and N'' is the transformed number of protons. The sixth column gives the ratio of the measured to the expected transformed count. The seventh column indicates the absence or presence of magnetic storms and Forbush decreases.

The data for May 4 and September 3, 1960 show satisfactory agreement with the measurements and the transformation. However, for May 12, July 12, and July 15, 1959 during magnetic storms and Forbush decreases the single-counter result

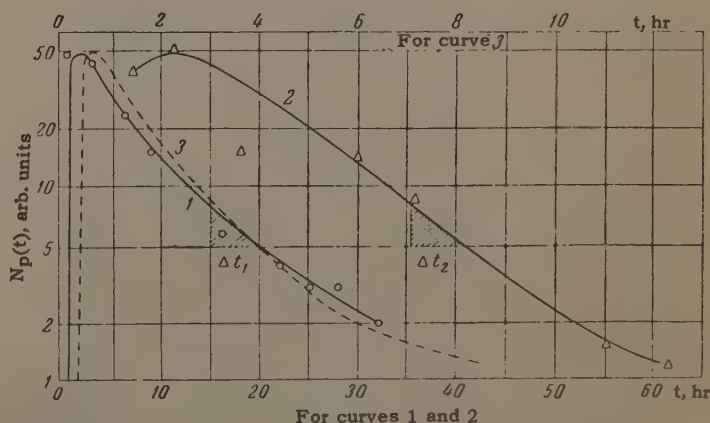


FIG. 5. Flare proton intensities during bursts in the stratosphere. Curve 1—May 4, 1960; curve 2—September 3, 1960; curve 3—burst registered February 23, 1956 at sea level in Chicago.

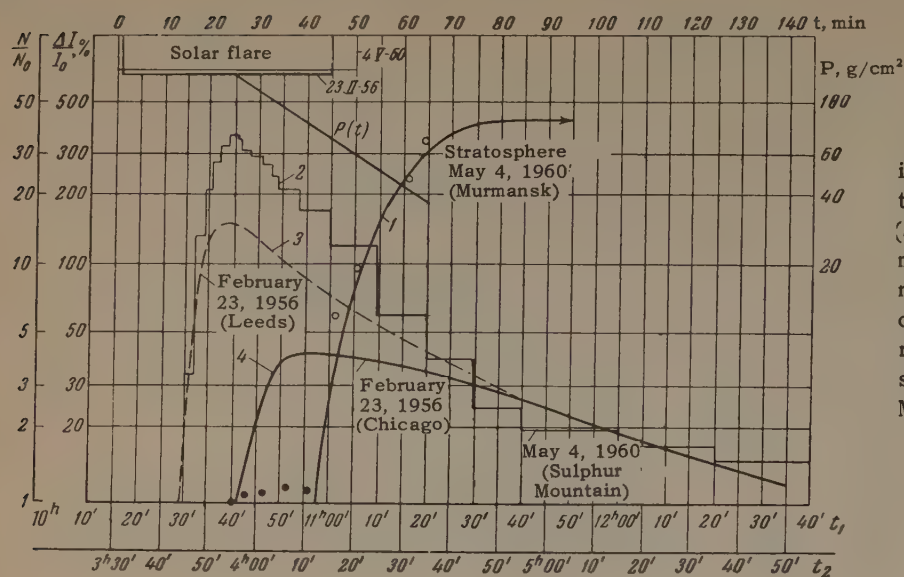


FIG. 6. Time dependence of the relative increase of cosmic ray intensity in the stratosphere (N/N_0 , curve 1) and at sea level ($\Delta I/I$, curves 2, 3, 4). t_1 — time of stratosphere measurements on May 4, 1960; t_2 — time of neutron monitor measurements; t — time after onset of solar flare. The rectangular steps represent the duration of the solar flares. The straight line represents pressure vs time on May 4, 1960.

was three times greater than the transformed (telescope) proton count for May 12, twice as large for July 12, and six times greater for July 15. Thus during a Forbush decrease the stratosphere is penetrated by particles with ranges under 7 mm Al in addition to protons. It is also possible that these short-range particles in the stratosphere are associated with outer Van Allen belt electrons.^[14]

TIME DEPENDENCE OF PRIMARY PROTON INTENSITY

The foregoing results furnish information regarding the time dependence $N_p(t)$ of primary proton intensity. Figure 5 shows $N_p(t)$ for May 4–5 and September 3–5, 1960. The onset of the corresponding solar flares was taken as $t = 0$. On May 4 the solar flare had the coordinates $\varphi = +10^\circ$, $l = +90^\circ$ and lasted from 1015 to 1117 UT. The flare of September 3 had the coordinates $\varphi = +17^\circ$, $l = -90^\circ$ and began at 0040 UT. Curve 3 represents $N_p(t)$ at Chicago for the event of February 23, 1956.^[16] The peaks of all three curves are seen to agree in height.

The errors in the data represented by Fig. 5 resulted mainly from the extrapolation of the measurements to the top of the atmosphere, and for most points do not appear to exceed 20–30%.

The curves in Fig. 5 exhibit segments of approximately constant intensity lasting 3–5 hrs for May 4 and 10–20 hrs for September 3. The longer duration of the September burst compared with the May event is represented by the flat segment of curve 2. The slopes of $N_p(t)$ on the descending portions of the curves are similar ($\Delta t_1 \approx \Delta t_2$ for the shaded triangles in the figure).

This is in accordance with proton diffusion in interplanetary space. The decrease of proton intensity at any given energy depends only on the medium of propagation. Here we must consider that interplanetary space contains scattering centers of cosmic rays in the form of magnetic clouds.^[15, 16]

On this basis it is interesting to compare the data for $N_p(t)$ in the stratosphere for primary protons above 0.1 Bev with the data for 3–4 Bev protons registered in the event of February 23, 1956. Figure 5 shows that the rate of decrease in this instance (curve 3) is about four times faster than for curves 1 and 2. It will be shown subsequently that this difference resulted in the fact that the diffusion coefficients for primary protons at low and high energies differ by a factor of 4.

Our measurements of May 4 were obtained at 15 km. At this same time neutron monitors registered a $10 \pm 2\%$ peak of enhanced cosmic ray intensity at Prague^[17] and 8% at Leeds.^[18] Curve 3 in Fig. 1 represents the progress of the stratosphere burst, based on the altitude dependence of N_{12} . A sharp rise of N_{12} with decreasing pressure was observed when the pressure on the instruments was under 90 g/cm². The arrival of low-energy primary protons in the stratosphere must be assigned to this period.

For $P < 90$ g/cm² the measurements of N_{12} were converted to the initial intensity of protons above 100 Mev (the open circles in Fig. 6). For $P > 90$ g/cm² unconverted values of N_{12} are given (filled circles). Figure 6 also includes the data for May 4, 1960 at Sulphur Mountain^[19] and for February 23, 1956 at Leeds^[15] and Chicago.^[16] The latter data are compared with those in^[19] for $t = 115$ min. We observe that on May 4 and Febru-

ary 23 cosmic rays lagged approximately 15 min. behind the onset of the solar flare. Maximum cosmic ray intensities are reached in 8–10 min. Low-energy protons lag 25–30 min behind the arrival of high-energy protons. This is a short period of time, considering the fact that most protons in the atmosphere have velocities $\beta = 0.5$, while on the ground $\beta \approx 1.0$. It is therefore possible that on May 4, 1960 low-energy and high-energy protons were generated simultaneously on the sun.

DISCUSSION OF THE TIME DEPENDENCE $N_p(t)$

On the basis of the data for February 23, 1956 many authors have stated that for a long period following the peak of a burst the cosmic ray intensity is described by $A/t^{3/2}$, and that during this time the spatial distribution of primary protons is isotropic. Both effects were accounted for by the diminishing primary proton intensity registered on the ground as a result of diffusion in interplanetary space. On the basis of the fact that the buildup time of the burst of February 23, 1960 was very much shorter than its decay, Meyer, Parker, and Simpson^[15] have constructed the following model for the spatial distribution of scattering centers: The diffusing magnetic field is situated mainly outside of the earth's orbit and there are no scattering centers between the sun and the earth. We shall show below that the experimental results can be accounted for without recourse to this special model, as was pointed out by Dorman.^[20] We shall now attempt to describe our results in the stratosphere on the basis of proton diffusion in interplanetary space.

The differential equation of diffusion in a spherically symmetrical uniform space, $\partial N_p / \partial t = D \nabla^2 N_p$, with the injection of a number B of particles at the point $r = 0$ and time $t = t_0$ has the solution

$$N_p(R, t - t_0) = \frac{1}{8} B [\pi D (t - t_0)]^{-3/2} \exp [-R^2 / 4D (t - t_0)]. \quad (1)$$

Here $N_p(R, t - t_0)$ in our case is the primary-proton intensity as a function of time on the earth; R is the sun-earth radius, t is the observation time; t_0 is the flare onset time; and D is the diffusion coefficient

$$D = lv/3 \quad (2)$$

where l is the mean free proton path for scattering, and v is the proton velocity.

Efficient scattering of protons by magnetic clouds at the distance l requires the condition $H > p/300l$, where H is the magnetic field

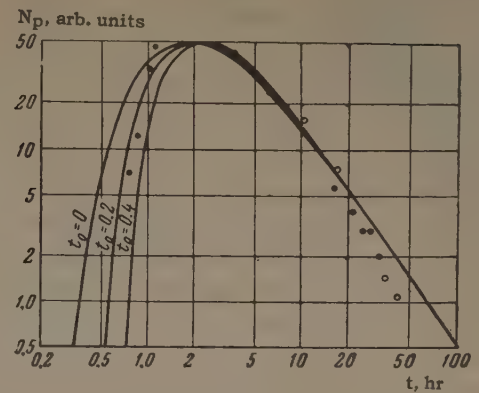


FIG. 7. Primary-proton intensity calculated from Eq. (1) for $D = 5.5 \times 10^{21}$ cm²/sec and $t_0 = 0, 0.2$ and 0.4 . ● — May 4, 1960; ○ — September 3, 1960.

strength in the clouds and p is the proton momentum.

The hypothesis that magnetic clouds are distributed uniformly around the sun is not only unsupported by any experimental data, but actually makes it impossible to account for some observations. However, this simplification is apparently justified as a means of explaining the principal features of the phenomena.

The behavior of $N_p(t - t_0)$ in (1) for $t - t_0 \ll R^2/4D$ is basically exponential. For this case it is extremely important to note the exact value of t_0 . The maximum of $N_p(t - t_0)$ appears at $t - t_0 = R^2/6D$. When $t - t_0 \gg R^2/4D$, we have

$$N_p(t - t_0) = B/8 [\pi D (t - t_0)]^{3/2}. \quad (3)$$

The constant B is determined experimentally from the maximum of $N_p(t)$. For the burst of May 4, $B = 2.5 \times 10^{32}$ protons, and the energy carried off by the protons was 6×10^{28} ergs.

The curves in Fig. 7 were plotted from Eq. (1) for three values of t_0 with $D_1 = 5.5 \times 10^{21}$ cm²/sec, and are compared with the data for May 4 and September 3. For May 4 time is measured from the onset of the solar flare. For the data of September 3 zero time is 20 hrs after the flare onset. This was done because the greater duration of the September 3 event resulted from the prolonged emission of protons directly from the source (Fig. 5). In order to have approximately identical initial conditions for diffusion in connection with both bursts, this period of time must, of course, be excluded. With D known we thus obtain from (2) the mean free path of low-energy protons, which is about one-tenth of the sun-earth radius. From the condition $H_1 > p/300l_1$ and $p_1 = 5 \times 10^8$ ev/c we obtain $H_1 > 1.5 \times 10^{-6}$ gauss.

From (3) and the recorded data along the descending segments of the curves in Fig. 5 we find

that the diffusion coefficient D_2 for 3–4 Bev protons is about four times larger than D_1 . The mean free path of high-energy protons is $l_2 = 2.2 \times 10^{12}$ cm. The magnetic field in clouds for the high-energy case is correspondingly $H_2 > p_2/300l_2 = 7.6 \times 10^{-6}$ gauss ($p_2 = 5 \times 10^9$ ev/c).

The data for l_1 and l_2 and the corresponding values of H_1 and H_2 lead to the conclusion that the density of magnetic clouds in space is inversely proportional to the magnetic field strength.

For $t < 1$ Fig. 7 shows the best agreement between the data and calculation for $t_0 = 0.3$, but the significance of this agreement is not clear. The cosmic ray burst at Sulphur Mountain was registered 15 min after the onset of the solar flare. Taking into account the time required for protons to reach the earth, we obtain a period 0–7 min or ~ 0.1 hr during which protons could begin to flow from the solar source. Assuming the simultaneous generation of low- and high-energy protons, we have $t_0 \sim 0.1$ hr. However, for $t_0 = 0.1$ hr the first two experimental points in Fig. 7 lie off the calculated curve. It is difficult to determine whether this means that slow protons are generated later ($t_0 = 0.3$ hr), or whether it results from neglecting the nonuniform spatial distribution of magnetic clouds, which are somewhat denser near the sun than near the earth. We note that the observed disagreement between the cosmic ray burst onset on the earth and the presumed beginning of proton flow from the sun also characterizes high-energy protons (February 23, 1956). It is reasonable to assume that this disagreement results from neglecting the higher magnetic cloud density near the sun.

For high values of t the experimental points lie below the straight line in Fig. 7. This result can also be attributed to lower cloud density at great distances outside of the earth's orbit.

ENERGY SPECTRUM OF HIGH ENERGY FLARE PROTONS

The low-energy protons registered in the stratosphere and the high-energy protons responsible for the cosmic ray spike at sea level on May 4, 1960 were clearly generated during the same solar flare. It remains to determine whether both high- and low-energy protons were generated by a single mechanism and therefore belong to a single energy spectrum.

Let us consider the data for the proton spectrum. We assume that the 10% peak cosmic ray increases registered by a neutron monitor at Prague on May 4 can be attributed to primary protons above 3 Bev

(the cutoff for primary protons at the latitude of Prague is 2.0 Bev). This increase corresponds to a flux of 0.5 primary protons per $\text{cm}^2\text{-min-sr}$. From stratosphere data the primary-proton intensity above 400 Mev was 30 protons per $\text{cm}^2\text{-min-sr}$. These results indicate that in the 0.4–3.0 Bev range the spectrum had the exponent $\gamma = 2.0$. Data for the February 23, 1956 event lead to about the same result.^[21] This is an improbably low value of γ , since the measured primary-proton energy spectrum at sea level usually has $\gamma = 5.0$. The low value of γ can possibly result from the incorrect assumption that low- and high-energy solar protons belong to a single spectrum. Independent generation mechanisms for low- and high-energy protons are also indicated by the stratosphere event of September 3, when the primary-proton spectrum practically coincided with that of May 4, although the amplitude was about twice as great, whereas there was almost no cosmic ray increase at sea level on September 3.

CONCLUSIONS

1. The energy spectra of primary protons from the sun were measured during the bursts of April 28, May 4, May 13, and September 3, 1960. The spectra were similar and underwent no essential change throughout the two to three day duration of each burst. It follows that solar generation of primary protons occurs during a time considerably shorter than the duration of the observed bursts. The proton generation mechanism also seems to be the same in all of these bursts. The integral proton energy spectrum as a function of their kinetic energy E_p can be described by a power law with the exponent $\gamma = 2.0$ for E_p from 100 to 400 Mev.

2. The energy spectrum of solar flare protons was moderated during a Forbush decrease ($\gamma \approx 5.5$). This effect combined with the simultaneous harder spectrum of cosmic rays of galactic origin during the Forbush decrease shows that solar corpuscular streams carrying frozen-in magnetic fields are carriers of solar-generated protons. In accounting for this effect we must assume the existence of magnetic traps in the solar corpuscular streams.

3. During a Forbush decrease the stratosphere is penetrated by other particles with ranges under 7 mm Al, in addition to protons. The origin of these short-range particles appearing in the stratosphere only during a Forbush decrease is not clear.

4. The time dependence of primary-proton in-

tensity in stratosphere bursts agrees satisfactorily with the theory of proton diffusion in interplanetary space containing magnetic clouds as scattering centers. Data were obtained characterizing the density and magnetic fields of magnetic clouds responsible for the scattering of ~ 0.2 -Bev protons in interplanetary space.

¹Charakhch'yan, Tulinov, and Charakhch'yan, JETP **39**, 249 (1960), Soviet Phys. JETP **12**, 179 (1961).

²Winckler, Bhavsar, and Peterson, Preprint, 1960.

³Ney, Winckler, and Freier, Phys. Rev. Letters **3**, 183 (1959).

⁴Charakhch'yan, Tulinov, and Charakhch'yan, JETP **38**, 1031 (1960), Soviet Phys. JETP **11**, 742 (1960).

⁵A. I. Kuz'min and G. V. Skripin, Trans. Yakutsk Branch, Acad. Sci. U.S.S.R. Phys. Ser. No. 3, 121 (1960).

⁶M. Kitamura and M. Kodama, Trans. International Conference on Cosmic Rays, July, 1959, Acad. Sci. U.S.S.R. Press, 1960, Vol. 4, p. 204.

⁷A. N. Charakhch'yan, Trudy MGG (Trans. IGY), in press.

⁸Arnoldy, Hoffman, and Winckler, J. Geophys. Research **65**, 3004 (1960).

⁹Fan, Meyer, and Simpson, Phys. Rev. Letters **5**, 269 (1960).

¹⁰J. A. Van Allen and J. Wei Ching-lin, J. Geophys. Research **65**, 2998 (1960).

¹¹A. M. Perry, Phys. Rev. **85**, 497 (1952).

¹²Bernardini, Booth, and Lindenbaum, Phys. Rev. **85**, 826 (1952).

¹³Meshcheryakov, Nurusev, and Stoletov, JETP **31**, 361 (1956), Soviet Phys. JETP **4**, 337 (1957).

¹⁴S. N. Vernov and A. E. Chudakov, loc. cit.^[6], Vol. 3, p. 17.

¹⁵Meyer, Parker, and Simpson, Phys. Rev. **104**, 768 (1956).

¹⁶L. I. Dorman, Variatsii kosmicheskikh lucheï (Cosmic Ray Variations), Gostekhizdat, 1957 (Eng. transl. by U.S. Air Force Technical Documents Liaison Office).

¹⁷Hladky, Kleczek, Krivsky, and Mokry, Nuovo cimento **17**, No. 4, 1960.

¹⁸M. Kadama, Preprint, 1960.

¹⁹Anderson, Chasson, Liwschitz, and Suda, J. Geophys. Research **65**, 3889 (1960).

²⁰L. I. Dorman, Nuovo cimento Suppl. **8**, 391 (1958).

²¹Charakhch'yan, Tulinov, and Charakhch'yan, Geomagnetizm i aërologiya (Geomagnetism and Aerology), in press.

Translated by I. Emin
131

PROTON-PROTON INTERACTION AT 9 Bev

V. A. KOBZEV, Yu. T. LUKIN, Zh. S. TAKIBAEV, G. R. TSADIKOVA, and E. V. SHALAGINA

Kazakh State University

Submitted to JETP editor April 27, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 747-751 (September, 1961)

The c.m.s. angular distributions of protons and π mesons emitted in emulsions in proton-proton collisions were measured. The proton angular distribution is symmetric and possesses a pronounced anisotropy. The π -meson distribution is symmetric and more isotropic. The angular dependence of the total energy in the c.m.s. has been determined for protons. It is shown that, on the average, protons expend about 74% of their energy in the c.m.s. on meson production. Within the limits of experimental error, the mean values of the transverse momentum and the total energy in the c.m.s. are the same for various multiplicities.

A large number of experiments^[1-3] have been carried out during the last two years to study the interaction of protons with nucleons at 9 Bev. In these experiments, the main features of pp collisions were revealed, and certain ideas on the nucleonic "dimensions" were also obtained. Unfortunately, the energy measurements and the identifications were carried out mainly for slow particles, and the results obtained by various authors are not always in agreement. Because of this, we have continued the study of pp interactions at 9 Bev energy.

An emulsion chamber consisting of layers of NIKFI-R emulsion was used. This chamber was irradiated by 9-Bev protons from the Joint Institute for Nuclear Research Synchrotron. The chamber was scanned along the tracks of primary protons using a magnification of 630 x. In all, 593 m of tracks were scanned, and 1609 nuclear interactions and scattering events at an angle $\theta \geq 5^\circ$ were found, which gives a mean free path for the interaction equal to 36.9 ± 0.9 cm. This value, within the limits of experimental error, is in agreement with the results obtained earlier by other authors.^[1,2]

The selection of inelastic pp interactions was based on the known criteria.^[1,2,4,5] It should be mentioned that among the number of disintegrations studied there was not a single case accompanied by the emission of at least one slow proton with range $l \leq 4$ mm.

Out of all the disintegrations found, 123 were considered as inelastic pp collisions. Their distribution with respect to the number of secondary charged particles is as follows:

Number of events:	2	4	6
Number of interactions, %:	44.7 ± 6.1	48.0 ± 6.2	7.3 ± 2.4

The mean number of charged particles per pp interaction was 3.25 ± 0.16 .

The secondary particles produced in inelastic pp collisions were identified by multiple Coulomb scattering and by the blob density along their tracks. The scattering and ionization measurements were carried out in an emulsion layer at least 20μ below the surface, along tracks having an angle of dip $\varphi \leq 5^\circ$. There were 147 such tracks.

In order to increase the measurement accuracy of both the multiple Coulomb scattering and the blob density, the tracks were followed in emulsion layers adjoining the ones used for the measurement.

Coulomb scattering was determined from the measurement of second differences of coordinates of three multiple cells.^[6-8] This method, as is well known, makes it possible to exclude false scattering for each separate track.

On tracks for which the number of independent second coordinate differences on double and quadruple cells is small, the quantity $p\beta$ was determined with account of false scattering averaged over the given emulsion layer. The mean values of second coordinate differences for false scattering D_f are as follows:

Cell length, μ :	500	1000	2000
D_f , μ :	0.141	0.221	0.411

To increase the accuracy of the estimate of $p\beta$ in all events, we have taken the relation between the

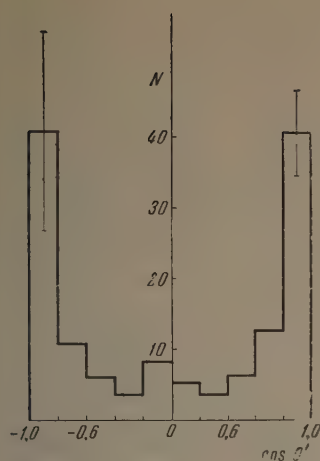
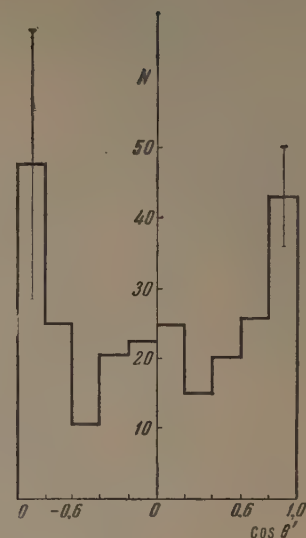


FIG. 1. Angular distribution of protons in the c.m.s.

 FIG. 2. Angular distribution of π mesons in the c.m.s.


second and third coordinate differences into account.

The particles were identified from the well-known dependence of ionization on particle velocity.^[9] The corresponding curve was verified in the region of velocities greater than 0.94 of the velocity of light by measurements of the blob density along 9-Bev proton tracks and along tracks of electron-positron pair particles with sufficiently high energy, produced by high-energy γ rays.

The identification of particles with $p\beta$ between 1.5 and 2.5 Bev is very difficult and not always unambiguous. In the present experiment, the particles belonging to this range were identified by the method proposed by Kalbach et al.^[4] As a result, 32% of the particles with $p\beta$ between 1.5 and 2.5 Bev were identified as protons, and the remainder as π mesons.

Using such an approach to the particle identification, the emission half-angles in the laboratory system (l.s.) for protons and π mesons in the doubtful energy range were found to be, on the average, equal to $\theta_p = (6 \pm 1)^\circ$ and $\theta_\pi = (18 \pm \frac{3}{9})^\circ$. Momentum and angular distributions in the l.s. for well-identified protons and π mesons have gaps in the range corresponding to $1.5 \leq p\beta \leq 2.5$ Bev which disappear if we include protons and π mesons from the doubtful range in the distribution.

The identification showed that, out of all 147 measured tracks, 70 belong to protons, 74 to π mesons, and 3 to K mesons. Taking into account the geometrical correction introduced by a method

similar to that of Bogachev et al.,^[10] the number of protons was found equal to 141, that of π mesons 255, and that of K mesons 26. The mean number of protons per interaction is $\bar{n}_p = 1.2 \pm 0.2$, and the mean number of charged π mesons is $\bar{n}_\pi = 1.9 \pm 0.2$.

The angular distribution of secondary protons in the c.m.s. is shown in Fig. 1. It is characterized by symmetry with respect to the angle $\theta' = 90^\circ$, and by sharp anisotropy. The emission half-angle for protons emitted forwards equals $(33 \pm 7)^\circ$. The angular distribution of π mesons in the c.m.s. (Fig. 2) is also symmetrical with respect to the plane dividing the forward and backward hemispheres, but is characterized by a markedly smaller anisotropy. The emission half-angle obtained by the same method as for protons equals $(51 \pm \frac{7}{11})^\circ$.

The other main characteristics of secondary protons and mesons are shown in the table, where the values of the total energy E' refer to the c.m.s. It can be seen from the table that the mean proton energy emitted forwards in the c.m.s. is identical with the mean proton energy emitted backwards in the c.m.s. The same is also observed for the π meson energies. However, the energy values obtained by us both for mesons and protons differ from the results of^[3]. We have obtained somewhat higher π meson energies, while the proton energy is lower than that of^[3].

	$\bar{n}$	$\bar{p}_1$, Mev/c	$\bar{E}'$, Mev		
			Particles emitted forward	Particles emitted backwards	All particles
Protons	1.20 ± 0.14	437 ± 52	1311 ± 53	1257 ± 77	1280 ± 42
π mesons	1.90 ± 0.20	314 ± 37	509 ± 59	527 ± 132	518 ± 61
K mesons	0.17 ± 0.10	—	—	920 ± 240	920 ± 240

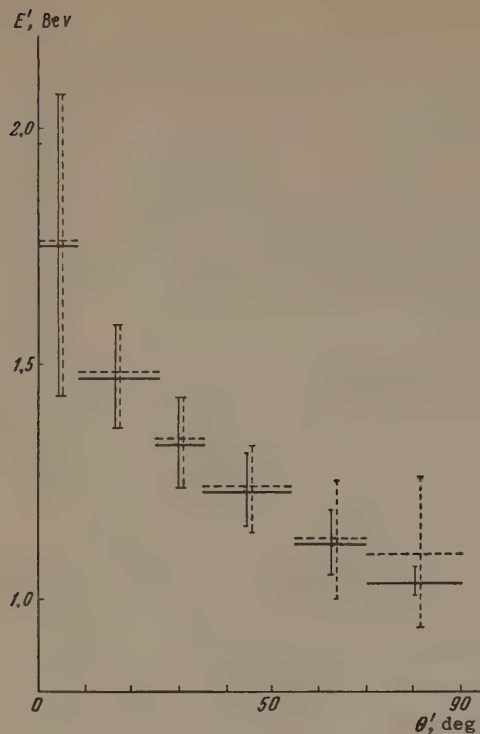


FIG. 3. Variation of the total proton energy with the angle of emission in the c.m.s.

We believe that one of the possible reasons for this discrepancy may lie in the different approach to particle identification in the doubtful region $1.5 \leq p\beta \leq 2.5$ Bev. As a rule, protons having $p\beta$ between 1.5 and 2.5 Bev in the l.s. are emitted in the c.m.s. with small energies and at large angles to the direction of motion of primary particles. It is therefore natural to expect that in the c.m.s. their mean energy will be lower, and the mean angle of emission greater, than for protons with values $p\beta < 1.5$ and $p\beta > 2.5$ Bev.

This is illustrated by Fig. 3, which represents the dependence of the mean total proton energy in the c.m.s. on the angle of emission in the c.m.s. The angle of emission for a particle going forwards in the c.m.s. is measured from the direction of motion of the primary proton, while, for particles propagating backwards in the c.m.s., it is measured from a line placed at 180° to the direction of

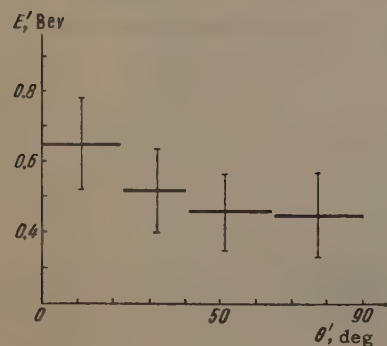


FIG. 4. Dependence of the total energy of π mesons on the angle of emission in the c.m.s.

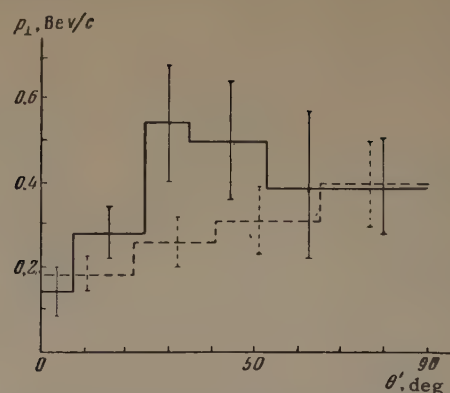


FIG. 5. Dependence of the transverse momentum of protons (solid line) and π mesons (dashed line) on the angle of emission in the c.m.s.

the incident particles. The solid lines correspond to the case where all protons, regardless of the value of $p\beta$ in the l.s., were included in the calculation, and the dotted lines to the case where protons with $p\beta$ between 1.5 and 2.5 Bev were excluded from the calculation. It can be seen from Fig. 3 that the mean proton energy decreases with increasing emission angle θ' .

An analogous dependence of the total energy of π mesons on the angle of emission in the c.m.s. is shown in Fig. 4. The c.m.s. angular distribution of transverse proton and π -meson momenta is shown in Fig. 5. The corresponding mean transverse momentum, both for protons and π mesons, depends weakly on the angle of emission of the particle in the c.m.s., with the exception of the range of small angles where the momentum conservation law begins to be felt.^[11]

The calculation of the correlation coefficients between the total momentum and the angle of emission θ' , and between the transverse momentum and the angle θ' , has shown that, for protons, the total momentum varies more with the angle than the transverse momentum.

A similar calculation carried out for the π mesons has led to the conclusion that the dependence of the total and of the transverse momenta on the emission angle of the particle in the c.m.s. is of similar magnitude.

The inelasticity factors determined from nucleon and π meson energies in the c.m.s. are equal to 0.74 ± 0.09 and 0.68 ± 0.15 respectively. If we assume that the difference in the inelasticity factor obtained by different methods is due to the underestimate of the proton energy, then it follows from the proton energy balance in the c.m.s. that the mean proton energy is $\bar{E}' = 1.37 \pm 0.05$ Bev.

A calculation carried out under the assumption that the mean energy of neutral and charged mesons

is the same shows that the mean number of neutral π mesons per interaction decreases from 2.7 (in stars with two charged particles) to zero (in events with $n_s = 6$). The total average number of π mesons per interaction is 3 or 4 for stars with different multiplicity, and the fraction of π^0 mesons on the average amounts to 0.4 ± 0.1 of the total number of mesons.

From the calculations carried out for events in which the momenta of all charged particles were measured, it follows that ~ 3.7 π mesons are produced on the average, in the reactions of the type $p + p \rightarrow p + p + n\pi^0$ and $p + p \rightarrow p + p + \pi^+ + \pi^- + n\pi^0$.

The result that, for protons and π mesons, the mean values of transverse momentum and of the total energy in the c.m.s. do not vary with multiplicity within the limits of experimental error is in agreement with the results obtained by other authors.^[3, 11, 12]

¹Bogachev, Bunyatov, Gramenitskii, Lyubimov, Merekov, Podgoretskii, Sidorov, and Tuvdendorzh, JETP **37**, 1225 (1959), Soviet Phys. JETP **10**, 872 (1960).

²Bogachev, Bunyatov, Merekov, and Sidorov, DAN SSSR **121**, 617 (1958), Soviet Phys.-Doklady **3**, 785 (1958).

³Wang, Vishki, Gramenitskii, Grishin, Dalkhazhav, Lebedev, Nomofilov, Podgoretskii, and

Strel'tsov, JETP **39**, 57 (1960), Soviet Phys. JETP **12**, 41 (1960).

⁴Kalbach, Lord, and Tzao, Phys. Rev. **113**, 330 (1959).

⁵N. G. Birger and Yu. A. Smorodin, JETP **36**, 1159 (1959), Soviet Phys. JETP **9**, 823 (1959).

⁶Chasnikov, Takibaev, and Boos, PTE (Instruments and Exptl. Techniques), No. 1, 54 (1959).

⁷Lukin, Takibaev, and Shalagina, JETP **39**, 1074 (1960), Soviet Phys. JETP **12**, 747 (1960).

⁸I. Ya. Chasnikov, Dissertation, Kazakh State University, 1961.

⁹W. H. Barkas and D. M. Young, UCRL-2579 (1954).

¹⁰Bogachev, Bunyatov, Vishki, Merekov, Sidorov, and Yarba, JETP **38**, 432 (1960), Soviet Phys. JETP **11**, 317 (1960).

¹¹Boos, Botvin, Takibaev, Pavlova, and Chasnikov, JETP **41**, 993 (1961), Soviet Phys. JETP **14**, in press.

¹²Belyakov, Wang, Glagolev, Dalkhazhev, Lebedev, Mel'nikova, Nikitin, Petrzhilka, Sviridov, Suk, and Tolstov, JETP **39**, 937 (1960), Soviet Phys. JETP **12**, 650 (1960).

Translated by H. Kasha

132

ANOMALOUS DOPPLER EFFECT IN A PLASMA

M. A. GINTSBURG

Submitted to JETP editor June 27, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 752-755 (September, 1961)

Excitation of electromagnetic waves in a plasma by a superluminal ion beam is considered by taking into account the motion of the ions in the plasma. The geophysical significance of this problem is pointed out.

THE radiation of an oscillator moving in a magnetoactive plasma with superluminal velocity has been investigated by a number of authors^[1-6] and the anomalous Doppler effect arising from such motion has been considered. In the present work, additional account is taken of the motion of the ions of the plasma, and certain geophysical applications of the results are indicated.

1. Let an ion of mass M_1 move in a plasma along the external magnetic field H with a velocity u . Inasmuch as it simultaneously rotates about the lines of force of the field H , it can be regarded as an oscillator with a set of natural frequencies $\omega_s = s\Omega_1$ ($s = 1, 2, \dots$; Ω_1 is the Larmor frequency of the ion). Substituting in the formula for the Doppler effect

$$\omega = \frac{\Omega_1}{(un/c) \cos \theta - 1}$$

the well known^[7] expression for the index of refraction of the extraordinary wave (with account of the ions), we find the following expression for the dimensionless frequency $\eta = \omega/\Omega$ for the case $\omega < \omega_H < \omega_0$, e and $\theta = 0$ (i.e., for propagation along the field):

$$u/v_A = (1 + Q/\eta) \sqrt{(1 + \eta)(1 - \alpha\eta)}, \quad (1)$$

where m — mass of the electron, M — mass of the plasma ion, $\alpha = m/M$, $Q = M/M_1$, N — concentration of ions in the plasma, $v_A = H/\sqrt{4\pi NM}$ — the Alfvén velocity.

The approximate values of the roots of Eq. (1) are:

$$\eta_1 = v_A/u, \quad \eta_2 = (u/v_A)^2, \quad \eta_3 = 1/\alpha - (u/v_A)^2.$$

2. The plot of the right side of Eq. (1) is shown in the drawing (curve 1). From this plot, we obtain the following results:

1) When $u > (1/2) v_A \sqrt{M/m}$, the ion excites a wave of only one frequency F_1 .

2) When $1/2 v_A \sqrt{M/m} > u > 2.6 v_A$, the ion excites waves in the three frequency ranges F_1 , F_2 and F_3 .

3) When $u < 2.6 v_A$, the ion excites (in the case of superluminal motion) a wave only in the region of gyromagnetic resonance of the electrons ($\omega \sim \omega_H$). When $\omega \sim \omega_H$ it is also necessary to take into account the collision-free resonance absorption and to compare the decrement associated with it with the increment brought about by the ion (or by the ion beam).

The essentially new effect, which follows from Eq. (1) and the figure, is the excitation by the ion of electromagnetic waves with a frequency less than the ion Larmor frequency, i.e., the excitation of magnetohydrodynamic waves. It is also of interest that the ion excites electronic plasma vibrations as well (the range F_3).*

3. For waves propagating at an angle θ to the field H , the condition for excitation is similar to (1):

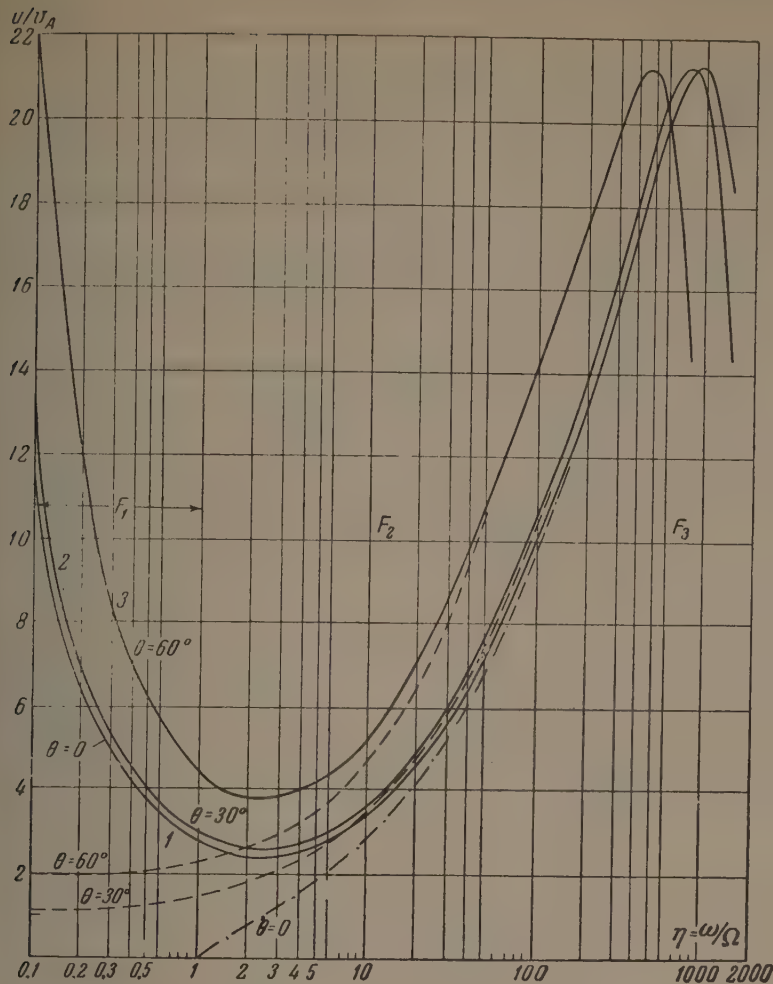
$$\frac{u}{v_A} = \left(1 + \frac{Q}{\eta}\right) \frac{1}{n(\theta, \eta) \cos \theta} \frac{c}{v_A}, \quad (2)$$

but the dependence of the index of refraction n on θ and η has a much more complicated form.^[7] The right side of Eq. (2) is shown in the figure by the curves 2 and 3 ($\theta = 30^\circ$ and $\theta = 60^\circ$).

There exist three cones of radiation, corresponding to the three frequency ranges F_1 , F_2 , and F_3 . In the first of these (F_1) the minimum frequency is propagated along the field, and the radiated frequency increases with increase in angle θ ($\eta_{\max} \approx \pm 1$). In the second cone (F_2), the frequency is inversely proportional to the angle. For the third cone (F_3), the elementary theory developed here is insufficient; as was shown above, one must also take into account the spatial dispersion in the plasma.

For $Q = 0$, Eq. (2) is the condition for Cerenkov radiation (the corresponding curves are shown as dashed lines in the figure).

*The correlation predicted in the present paper between the radiation in the ranges F_1 , F_2 and F_3 was discovered experimentally in;^[8] the radiation frequencies agree with Eq. (1).



Condition for radiation: solid curves – for the anomalous Doppler effect, dash-dot curve – for the normal Doppler effect, dotted curve – for the Cerenkov effect.

In the range of frequencies $\alpha\eta^2 < 1$ ($\omega < \sqrt{\omega_H \Omega}$) the following approximate expression for n was used in constructing the curves of the drawing:*

$$n = \left\{ \frac{1 + \cos^2 \theta + [(1 + \cos^2 \theta)^2 - 4 \cos^2 \theta (1 - \eta^2)]^{1/2}}{2(1 - \eta^2) \cos^2 \theta} \right\}^{1/2} \frac{c}{v_A} \quad (3)$$

Obviously the low-frequency vibrations also excite relativistic particles, for example, relativistic protons of the internal radiation belt. For relativistic particles, the term Q/η in (1), (2) must be replaced by $(Q/\eta)(1 - u^2/c^2)^{1/2}$.

4. The sun periodically emits streams of protons; these are the solar corpuscular streams. On the other hand, it is known that the magnetic field of the earth undergoes vibrations of small amplitude—the so-called short period variations with frequencies of 10^{-2} –10 cps. In the present article we advance the hypothesis that one of the possible mechanisms of the short period variations is the radiation of the protons of the solar corpuscular streams, due to their Larmor rotation around the lines of force of the earth's magnetic field (cyclotron instability of the stream in its

superluminal motion in the earth's plasma). The vibrations with frequencies of the order of a fraction of a cycle per second correspond to the magnetohydrodynamic branch of the radiation (the range F_1 in the drawing).*

MacArthur^[10] and Murcay and Pope^[11] explained the radiation of the solar corpuscular streams as being an incoherent cyclotron radiation along the field ($\theta = 0$) of ions moving in the earth's plasma. In contrast with our assumption, the forward motion of the ions is assumed by them in this case to be subluminal, $u < c/n$, and consequently the condition (1) is replaced by the expression

$$u/v_A = (1 - Q/\eta) \sqrt{(1 + \eta)(1 - \alpha\eta)},$$

*For the Alfvén wave [which corresponds to the plus sign in front of the root in (3)], the condition for cyclotron excitation is similar to (2): when $\eta \ll 1$, we have $\eta \approx v_A/u$ ($\theta \neq 0$), i.e., waves of a single frequency are generated at all angles to the field. It is also important that the group velocity of the Alfvén waves is parallel to the magnetic field for any θ . As a result, the amplitude of the Alfvén wave at the earth's surface is much larger than the amplitude of the oppositely polarized magnetoacoustic wave. The polarization observed in^[9] can be explained in this manner.

*Equation (3) was obtained independently by V. D. Shafranov.

which is shown in the drawing by the dash-dot curve. But it follows from the general expression obtained by Éidman^[4] for the radiation of a particle moving in a gyrotropic medium (supplemented by account of the motion of the plasma ions) that when $u < c/n$ the ion does not radiate along the field: the Doppler-effect equation for the frequency has no roots for the ordinary wave, while there are roots for the extraordinary wave (they were found by MacArthur^[10]), but the intensity of the radiation is equal to zero at $\theta = 0$.

5. We now consider the radiation not of an isolated particle but of an unbounded ion beam. Assuming for the beam a shifted Maxwellian distribution:

$$f(v) = \text{const} \cdot \exp \{-S^{-2} [v_x^2 + v_y^2 + (v_z - u)^2]\},$$

where $S = (2kT)^{1/2} M^{-1/2}$ — thermal velocity, we calculate the index of refraction at $\theta = 0$ from the general formula for the permittivity of a hot plasma (see, for example,^[12]):

$$n^2 = 1 + \frac{i\sqrt{\pi}}{(\omega + i\gamma)^2 k} \sum_{l=1}^4 \frac{\omega_{0l}^2}{S_l} (\omega + i\gamma - ku_l) W(p_l), \quad (4)$$

where k — wave number, γ — increment ($E, H \sim e^{\gamma t} e^{i(kz - \omega t)}$),

$$p_l = \frac{\omega \mp \omega_{H,l} + i\gamma - ku_l}{kS_l},$$

$$W(p) = e^{-p^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^p e^{t^2} dt \right).$$

The sum in (4) is carried out over the electrons and ions of the plasma and the beam ($l = 1, 2, 3, 4$).

Assuming the beam to be hot and the plasma cold, we find from the equation $k^2 = (\omega + i\gamma)^2 n^2 / c^2$ the following expression for the increment:

$$\frac{\gamma}{\omega} = - \frac{\omega - ku}{kS} \frac{\omega_0^2}{\omega^2} \frac{1}{2\varepsilon_{p1} + \omega \partial \varepsilon_{p1} / \partial \omega}.$$

Equation (5) is obtained for $|p_4| \ll 1$ (that is, $\omega - ku \pm \Omega_1 = 0$, cyclotron instability), for low beam density ($\gamma/kS \ll 1$), and for not too high a temperature ($\omega/kS > 1$). In (5),

$$\varepsilon_{p1} = (c/v_A)^2 [(1 + \eta)(1 - \alpha\eta)]^{-1}$$

is the permittivity of the cold plasma, ω_0 — plasma frequency of the beam.

In the case of an anomalous Doppler effect ($\omega - ku + \Omega_1 = 0$, superluminal motion), it follows from (5) that $\gamma > 0$, the beam is unstable, the amplitude of the wave increases. Conversely, for the normal Doppler effect (i.e., for $\omega - ku - \Omega_1 = 0$, subluminal motion), it follows from (5) that $\gamma < 0$, i.e., in this case the buildup is replaced by damping.

The author is grateful to V. D. Shafranov and R. Z. Sagdeev for discussions.

¹K. A. Barsukov, JETP **36**, 1485 (1959), Soviet Phys. JETP **9**, 1052 (1959).

²K. A. Barsukov and A. A. Kolomenskii, ZhTF **29**, 954 (1959), Soviet Phys.-Tech. Phys. **4**, 868 (1959).

³V. L. Ginzburg and V. Ya. Éidman, Izv. VUZov, (News of the Colleges, Radiophysics) Radiofizika **2**, 331 (1959).

⁴V. Ya. Éidman, JETP **34**, 131 (1958) and **36**, 1335 (1959), Soviet Phys. JETP **7**, 91 (1958) and **9**, 947 (1959).

⁵V. V. Zhelznyakov, Izv. VUZov, Radiofizika **2**, 14 (1959).

⁶V. L. Ginzburg, UFN **69**, 538 (1959), Soviet Phys.-Uspekhi **2**, 874 (1960).

⁷Al'pert, Ginzburg, and Feinberg, Rasprostranenie radiovoln (Radio Wave Propagation), Gostekhizdat, 1952, Sec. 63.

⁸Aaron, Gustafsson, and Egeland, Nature **185**, 148 (1960).

⁹M. Sugiura, Phys. Rev. Lett. **6**, 255 (1961).

¹⁰J. MacArthur, Phys. Rev. Lett. **2**, 491 (1959).

¹¹W. Murcray and J. Pope, Phys. Rev. Lett. **4**, 5 (1960).

¹²R. Z. Sagdeev and V. D. Shafranov, Proc. Second Geneva Conference on the Peaceful Uses of Atomic Energy (Russ. ed.), vol. 1, Atomizdat, 1959, p. 202.

Translated by R. T. Beyer
133

ON THE THEORY OF SPIN-LATTICE RELAXATION IN LIQUID SOLUTIONS OF ELECTROLYTES

K. A. VALIEV and M. M. ZARIPOV

Kazan' Pedagogical Institute

Submitted to JETP editor January 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 756-761 (September, 1961)

Spin-lattice relaxation of hydrated metal ions in aqueous solutions due to the interaction between the ion spin and the internal vibrations of the complex ion is investigated by the method of the theory of random processes.

1. INTRODUCTION

THIS paper is devoted to the calculation of the lifetime of the spin of a paramagnetic ion in a liquid solution in a given energy level. We base our discussion on the generally accepted notion that in solution a paramagnetic ion M forms together with the nearest diamagnetic particles X_i a stable paramagnetic complex ion whose vibrations can be characterized by a set of normal coordinates Q_j . The oscillations of Q_j produce variations of the electric field due to the particles X_i at the position of the paramagnetic ion at the center of the complex ion and interact via the spin-orbit coupling of the ion ($\mathcal{H}_{SO} = \lambda \mathbf{L} \cdot \mathbf{S}$) with the spin $\mathbf{S}$ of particle M . On the other hand, the oscillations of Q_j interact with the Brownian motion of the particles of the liquid surrounding the complex ion, giving rise to a set $Q_j(t)$ of independent narrow-band random Gaussian processes; the properties of the latter have been studied in detail.^[1-3] Although the energy $\hbar\omega_{MM'}$ released in the reorientation of the spin is small compared to the energy intervals $\hbar\omega_j$ between the sublevels of the oscillators Q_j , the random nature of the variation of the coordinates Q_j determines the possibility of the transfer of the spin energy of the ion M to the oscillations of Q_j describing the complex ion. We shall treat the coordinates Q_j semi-classically; a quantum mechanical treatment of the interaction of the oscillators Q_j with the medium yields results which are basically in agreement with the classical discussion.^[3]

The energy of interaction of the spin $\mathbf{S}$ of the central ion of the complex with the oscillations of Q_j can be represented in the general case in the form of the series

$$\mathcal{H} = \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \dots = \sum_i P_i(\mathbf{S}) Q_i + \sum_{ij} P_{ij}(\mathbf{S}) Q_i Q_j + \dots, \quad (1)$$

where $P_j(\mathbf{S})$, $P_{ij}(\mathbf{S})$ are quadratic functions in the components of the vector $\mathbf{S}$ (S_x , S_y , S_z). If x , y , z , define the laboratory system of coordinates, then the coefficients $P_{ij}(S_x, S_y, S_z)$ will, under rotation of the complex ion, turn out to be functions of randomly varying angles specifying the orientation of the complex ion with respect to the x , y , z system.

In the paper by one of the authors together with Al'tshuler^[4] a calculation was given of the rate of relaxation of the spin $\mathbf{S}$ due to the terms linear in Q_j in the expansion of the spin energy. Analysis shows, however, that a more effective relaxation is induced by the terms in the expansion of $\mathcal{H}^{(2)}$ which are quadratic in the Q_j .^{*} The same conclusion was obtained independently of us by Aleksandrov and Zhidomirov^[5] who considered the relaxation of the spin of a free radical due to its interaction with intramolecular oscillations. Therefore, for the analysis of this problem we refer the reader to the paper by Aleksandrov and Zhidomirov.^[5]

We shall evaluate the probability of relaxation transitions between the spin energy sublevels by means of the perturbation theory formula

$$\omega_{M,M'} = \hbar^{-2} \sum_{i,j} |\langle M | P_{ij}(\mathbf{S}) | M' \rangle \overline{Q_i Q_j}|^2 P_{ij}(\omega_{M,M'}), \quad (2)$$

^{*}Although the quadratic terms in the expansion give rise to a more rapid spin relaxation than the linear terms, a satisfactory description of the experimental results was achieved in^[4] because the expression $\exp(-\lambda_j|\tau|)$ describing the correlation of the amplitudes $A_j(t)$ of the processes $Q_j(t) = A_j(t) \cos(\omega t + \varphi_j)$ was chosen for the correlation function of the random processes $Q_j(t)$.

where $\rho_{ij}(\omega_{M,M'})$ is the spectral density of the perturbing energy $\mathcal{H}^{(2)}$ at the transition frequency $\omega_{M,M'}$; the bar above the expression denotes averaging over the arguments of $\mathcal{H}_{MM}^{(2)}$. By treating $\mathcal{H}^{(2)}$ as a product of independent random processes $\langle M | P_{ij}(S) | M' \rangle$ and $Q_i Q_j$ we evaluate the density $\rho(\omega_{M,M'})$ by means of the formula^[1,2]

$$\rho(\omega) = \int_{-\infty}^{+\infty} \exp(i\omega\tau) g(\tau) d\tau, \quad (3)$$

where $g(\tau)$ is the product of the correlation functions of the corresponding independent processes normalized to $g(0) = 1$. On assuming for the correlation functions the expressions^[1,2]

$$g_{Q_i Q_j}(\tau) = \exp(-(\lambda_j + \lambda_i)|\tau|) \cos \omega_i \tau \cos \omega_j \tau,$$

$$g_{Q_j^2}(\tau) = \frac{1}{3} + \frac{1}{3} \exp(-2\lambda_j|\tau|) + \frac{1}{3} \exp(-2\lambda_j|\tau|) \cos 2\omega_j \tau,$$

$$g_{P(S)}(\tau) = \exp(-|\tau|\tau_r^{-1}), \quad \tau_r = \frac{4}{3} \pi \eta a^3 / kT, \quad (4)$$

we obtain for the densities $\rho_{ij}(\omega)$ and $\rho_{jj}(\omega)$

$$\begin{aligned} \rho_{ij}(\omega) &= \frac{1}{2} \frac{\lambda_i + \lambda_j + \tau_r^{-1}}{(\omega_i + \omega_j - \omega)^2 + (\lambda_i + \lambda_j + \tau_r^{-1})^2} \\ &+ \frac{1}{2} \frac{\lambda_i + \lambda_j + \tau_r^{-1}}{(\omega_i + \omega_j + \omega)^2 + (\lambda_i + \lambda_j + \tau_r^{-1})^2} \\ &+ \frac{1}{2} \frac{\lambda_i + \lambda_j + \tau_r^{-1}}{(\omega_i - \omega_j - \omega)^2 + (\lambda_i + \lambda_j + \tau_r^{-1})^2} \\ &+ \frac{1}{2} \frac{\lambda_i + \lambda_j + \tau_r^{-1}}{(\omega_i - \omega_j + \omega)^2 + (\lambda_i + \lambda_j + \tau_r^{-1})^2}, \\ \rho_{jj}(\omega) &= \frac{2}{3} \frac{\tau_r^{-1}}{\tau_r^{-2} + \omega^2} + \frac{2}{3} \frac{2\lambda_j + \tau_r^{-1}}{\omega^2 + (2\lambda_j + \tau_r^{-1})^2} \\ &+ \frac{2\lambda_j + \tau_r^{-1}}{(2\lambda_j + \tau_r^{-1})^2 + (\omega_j + \omega)^2} + \frac{2\lambda_j + \tau_r^{-1}}{(2\lambda_j + \tau_r^{-1})^2 + (\omega_j - \omega)^2}. \end{aligned} \quad (5)$$

The parameters λ_j appearing in (4) and (5) can be interpreted as the probability for the dissipation of the vibrational energy of the complex ion. In accordance with^[6], $\lambda_j \sim \tau_c^{-1}$ for $\omega_j \tau_c \gg 1$, while the value of λ_j is $\lambda_j \sim 10^{12} \text{ sec}^{-1}$. The frequencies ω_j are of the order $\sim 10^2 \text{ cm}^{-1}$. Therefore, for $\omega_{M,M'} \ll \lambda_j$ we obtain $\rho_{jj}(\omega) \approx 10^{-11} \text{ sec}$, $\rho_{ij}(\omega) \approx 10^{-13} \text{ sec}$. In the case when the coordinates Q_i and Q_j are degenerate ($\omega_i = \omega_j$), we have $\rho_{ij} \approx \rho_{jj}$.

2. RELAXATION IN AQUEOUS SOLUTIONS OF Cr^{+3} SALTS

The energy of interaction $\mathcal{H}^{(2)}$ of the magnetic electrons of the ion with the vibrations of Q_j has the form*

*The evaluation of $\mathcal{H}^{(2)}$ has been carried out for us by A. M. Leushin, to whom the authors express their sincere gratitude.

$$\begin{aligned} \mathcal{H}^{(2)} &= W_1(2Q_1^2 + 2Q_2^2 + 2Q_3^2 - Q_4^2 - Q_5^2 - Q_6^2) \\ &+ W_2'(2\sqrt{2}Q_1Q_2 + 2Q_2Q_3) + W_2''(2\sqrt{2}Q_1Q_3 + Q_2^2 - Q_3^2) \\ &+ W_3'(\sqrt{3}Q_5^2 - Q_6^2\sqrt{3}) + W_3''(2Q_4^2 - Q_5^2 - Q_6^2) \\ &+ 2W_4'Q_2Q_4 + W_4''(Q_2Q_5 + \sqrt{3}Q_3Q_5) + W_4'''(Q_2Q_6 \\ &- \sqrt{3}Q_3Q_6) + W_5'(2\sqrt{2}Q_1Q_4 + 2Q_3Q_4) + W_5''(2\sqrt{2}Q_1Q_5 \\ &+ \sqrt{3}Q_2Q_5 - Q_3Q_5) + W_5'''(2\sqrt{2}Q_1Q_6 - \sqrt{3}Q_2Q_6 \\ &- Q_3Q_6) + W_6'''Q_4Q_5 + W_6''Q_4Q_6 + W_6'Q_5Q_6, \end{aligned} \quad (6)$$

where

$$\begin{aligned} W_1 &= A - 20F(x^4 + y^4 + z^4 - 3r^4/5), \\ W_2' &= B\sqrt{3}(x^2 - y^2) - 20F\sqrt{3}(x^4 - y^4), \\ W_2'' &= B(r^2 - 3z^2) - 20F(x^4 + y^4 - 2z^4), \\ W_3' &= C\sqrt{3}(x^2 - y^2) + 11\sqrt{3}(x^4 - y^4), \\ W_3'' &= C(r^2 - 3z^2) + 11F(x^4 + y^4 - 2z^4), \\ W_4' &= 48F(x^3y - xy^3), \\ W_4'' &= 48F(x^3z - xz^3), \\ W_4''' &= 48F(yz^3 - zy^3), \\ W_5' &= \sqrt{3}Dxy + 16\sqrt{3}F(x^3y + xy^3), \\ W_5'' &= \sqrt{3}Dxz + 16\sqrt{3}F(x^3z + xz^3), \\ W_5''' &= \sqrt{3}Dyz + 16\sqrt{3}F(y^3z + yz^3), \\ W_6' &= Exy + 48F(x^3y + xy^3), \\ W_6'' &= Exz + 48F(x^3z + xz^3), \\ W_6''' &= Exz + 48F(y^3z + yz^3); \\ A &= -\frac{1}{2}ee'R^{-3}, \quad B = -\frac{3}{8}ee'R^{-5}(4 - 25r^2R^{-2}), \\ C &= \frac{3}{32}ee'R^{-5}(2 - 55r^2R^{-2}), \quad D = \frac{1}{2}ee'R^{-5}(4 - 15r^2R^{-2}), \\ E &= -\frac{3}{2}ee'R^{-5}(1 + 15r^2R^{-2}), \quad F = \frac{35}{64}ee'R^{-7}. \end{aligned} \quad (7)$$

Here $\mathbf{r}(x, y, z)$ are the coordinates of the magnetic electron of the ion; R is the equilibrium distance between the nucleus of the ion and the center of charge e' of the diamagnetic particle appearing in the complex ion.

The ^4F term of the free Cr^{+3} ion in the cubic field due to six particles X_i is split in such a way that the lowest orbital level turns out to be single; the remaining orbital levels are separated from the lower one by $\sim 10^{-4} \text{ cm}^{-1}$. The matrix element of the perturbation $\mathcal{H}^{(2)} + \lambda \mathbf{LS}$ connecting the two spin levels M and M' appears in the third order approximation and is equal to

$$\begin{aligned} \mathcal{H}_{M,M'}^{(2)} &= e_1 [(-\sqrt{3}Q_2Q_3 + \frac{33}{40}(Q_5^2 - Q_6^2))(S_y^2 - S_x^2) \\ &+ (3S_z^2 - S^2)(-\frac{1}{2}(Q_2^2 - Q_3^2) + \frac{11}{40}(2Q_4^2 - Q_5^2 - Q_6^2))] \\ &+ e_2 [Q_5Q_6\{S_xS_y\} + Q_4Q_5\{S_xS_z\} + Q_4Q_6\{S_yS_z\}], \\ e_1 &= -\frac{27}{R^3} \left(\frac{\lambda}{\Delta} \right)^2 \left(\frac{ee'}{R} \right) \frac{r^4}{R^4}, \quad e_2 = \frac{18 \cdot 38 \sqrt{3}}{175 R^3} \left(\frac{\lambda}{\Delta} \right)^2 \left(\frac{ee'}{R} \right) \left(\frac{r^2}{K^2} - \frac{5}{38} \frac{r^4}{R^4} \right) \end{aligned} \quad (9)$$

Here Δ is the total splitting of the ${}^4F$ term in the cubic field; figure brackets denote the symmetrized product: $\{ab\} = ab + ba$. The data of Van Vleck^[7] have been utilized to simplify the expressions for ε_1 and ε_2 . We also note that in (9) we have omitted products of the coordinates Q_j with different frequencies which, in accordance with Sec. 1, give no essential contribution to spin relaxation. Before evaluating the matrix element between the spin states M and M' the spin variables in (9) must be transformed to the laboratory system of coordinates with the z axis parallel to the constant field H_0 :

$$\begin{aligned} S_{x'} &= \alpha_\mu S_\mu, & S_{y'} &= \beta_\nu S_\nu, \\ S_{z'} &= \gamma_\kappa S_\kappa, & \mu, \nu, \kappa &= x, y, z. \end{aligned} \quad (10)$$

Here α_μ , β_ν , γ_κ are the direction cosines between the axes of the moving system of coordinates (corotating with the complex ion) and the laboratory system. In averaging $|\mathcal{H}'|^2$ over the sphere it is convenient to express α_μ , β_ν , and γ_κ in terms of the Eulerian angles; in this way we can easily obtain

$$\begin{aligned} \overline{(S_{z'}^2)_{M,M'}}^2 &= \frac{2}{15} \{S_x S_z\}_{M,M+1}^2; & \frac{8}{15} (S_x^2)_{M,M+2}^2, \\ \overline{(S_{x'}^2 - S_{y'}^2)_{M,M'}}^2 &= \frac{2}{15} \{S_x S_z\}_{M,M+1}^2; & \frac{16}{15} (S_x^2)_{M,M+2}^2, \\ \overline{\{S_{\mu'} S_{\nu'}\}}^2 &= \frac{2}{15} \{S_x S_z\}_{M,M+1}^2; & \frac{16}{15} (S_x^2)_{M,M+2}^2. \end{aligned} \quad (11)$$

By combining (2), (9), and (11) we obtain for the probability of the transition $M, M+2$

$$\begin{aligned} w_{M,M+2} &= (S_x^2)_{M,M+2}^2 \hbar^{-2} \{ \varepsilon_1 [\frac{6}{5} (\overline{Q_4^2} p_{22} + \overline{Q_3^2} p_{33}) + \frac{48}{5} \overline{Q_2^2} \overline{Q_5^2} p_{23} \\ &+ \frac{768}{260} \overline{Q_2^2} p_{22} + \frac{19}{5} (\frac{33}{40})^2 (\overline{Q_5^2} p_{55} + \overline{Q_6^2} p_{66})] + \frac{16}{5} \varepsilon_2 [\overline{Q_4^2} \overline{Q_5^2} p_{45} \\ &+ \overline{Q_5^2} \overline{Q_6^2} p_{56} + \overline{Q_4^2} \overline{Q_6^2} p_{46}] \}. \end{aligned} \quad (12)$$

Here^[8]

$$\overline{Q_j^2} = \frac{\hbar}{2\omega_j \mu_j} \text{cth} \frac{\hbar \omega_j}{2kT}, \quad \overline{Q_j^4} = \frac{3}{4} \frac{\hbar^2}{\omega_j^2 \mu_j^2} \text{cth}^2 \frac{\hbar \omega_j}{2kT}. \quad (13)^*$$

Similarly we can obtain the probability of the transition $M, M+1$; it is of the same order of magnitude.

In Eq. (12) we have, in accordance with (5), $\rho_{ij} \approx \rho_{jj} \approx 10^{-11}$ sec. We set $Q^4 = 1.5 \times 10^{-37}$ cm⁴; by utilizing the values for the other constants quoted in^[7] we have obtained $\varepsilon_1 = 1$, $\varepsilon_2 = 0.1$ and $W_{M,M+2} \approx 5 \times 10^7$ sec⁻¹, which is lower than the experimentally observed width ($\sim 5 \times 10^8$ sec⁻¹,^[9]) by an order of magnitude. It should be noted in this connection that we have not taken into account the contributions to the line width due to the parts of $\mathcal{H}^{(2)}$ diagonal with respect to the spin which describe the shifts of the spin levels due to the oscillations of Q_j . Moreover, in the perturbation

term $\mathcal{H}^{(2)}$ we have not retained the coordinates of the vibrations of $Q_7 \dots Q_{15}$ which are not symmetric under an inversion. We also note that the values of many of the parameters are known only up to an order of magnitude, and, therefore, the agreement between the calculated and the observed widths can be regarded as satisfactory.

Formula (12) gives a reasonably good representation of the temperature dependence of the width of the resonance line of the Cr^{+3} ions in an aqueous solution of $\text{Cr}(\text{NO}_3)_3$ where ΔH_{exp} is equal to 245, 190, 125, 107, and 105 oe respectively at temperatures^[9] of 303, 323, 373, 423, and 473° K. On taking ΔH_{theor} (303° K) = 245 oe we have obtained values of ΔH_{theor} equal to 200, 137, 106, and 89 oe corresponding to temperatures of 323, 373, 423, and 473° K for $W_j = 560$ cm⁻¹ and $E/k = 1250^\circ \text{K}$.

In the case of an anisotropic g -factor or of a Stark splitting of the spin sublevels of the ion ($S > 1/2$) a different relaxation mechanism is possible which leads to the relation^[10,11]

$$T_1^{-1} \sim \tau_r / (1 + \omega_{M,M'}^2 \tau_r^2). \quad (14)$$

In accordance with our calculation

$$T_1^{-1} \sim \text{cth}^2 (\hbar \omega_j / 2kT) \tau_r / (1 + \omega_{M,M'}^2 \tau_r^2). \quad (15)$$

As can be seen from (14) and (15), up to temperatures $T \sim \hbar \omega_j / 2k$ the dependence of T_1^{-1} on T in the theories of McConnell^[10] and McGarvey,^[11] and in the theory under discussion here, is practically identical. As the liquid is heated further the decrease in T_1^{-1} predicted by formula (15) will be slower than in the theories of McConnell and of McGarvey, and one can even expect an increase in T_1^{-1} at temperatures $T \gg \hbar \omega_j / 2k$. Apparently, this case is in fact realized in aqueous solutions of Cr^{+3} salts.

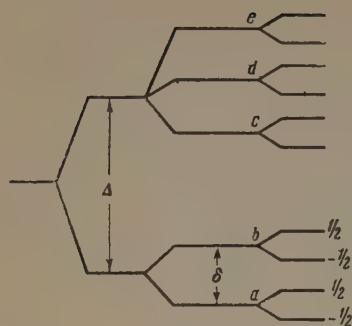
3. WIDTH OF THE RESONANCE LINE OF Cu^{+2} IONS IN AQUEOUS SOLUTION

The problem of the nature of the width of the resonance line of the Cu^{+2} ion in an aqueous solution deserves special consideration. The energy level diagram for a doubly ionized copper ion surrounded by six particles forming a slightly distorted octahedron is shown in the figure. The orbital levels are characterized by the following wave functions:^[12]

$$\begin{aligned} \psi_a &= (\psi_2 + \psi_{-2})/\sqrt{2}, & \psi_b &= \psi_0, & \psi_c &= (\psi_2 - \psi_{-2})/\sqrt{2}, \\ \psi_d &= (\psi_1 + \psi_{-1})/\sqrt{2}, & \psi_e &= (\psi_1 - \psi_{-1})/\sqrt{2}. \end{aligned} \quad (16)$$

The intervals Δ and δ respectively amount to $\sim 10^4$ and $\sim 10^3$ cm⁻¹. The spin sublevels arising

*cth = coth.



The progressive splitting of the fundamental ^2D energy level under the action of a strong cubic field, a weak tetragonal field, and a magnetic field.

when a constant magnetic field is applied are characterized by the quantum numbers $M = \pm 1/2$.

We have found that the width $\Delta\nu$ of the resonance line observed as a result of transitions a , $M = -1/2 \rightarrow a$, $M = +1/2$ is determined not by the spin-relaxation transitions a , $M = 1/2 \rightarrow a$, $M = -1/2$, excited by the combined action of $\mathcal{H}^{(2)}$, the Zeeman and the spin-orbit interactions of the ion, but by the relaxation transitions between the orbital sublevels a and b excited under the action of $\mathcal{H}^{(2)}$ alone with the interaction $\lambda \mathbf{L} \cdot \mathbf{S}$ taking no part.

Indeed, in accordance with the general principles of quantum mechanics the width $\Delta\nu$ of the resonance line of the transition a , $M = -1/2 \rightarrow a$, $M = +1/2$, is equal to the sum of the probabilities of relaxation transitions from each of the sublevels a , $M = \pm 1/2$ to all the other sublevels. The relaxation transitions between the orbital sublevels a , $M = 1/2 \rightarrow b$, $M = 1/2$ and a , $M = -1/2 \rightarrow b$, $M = -1/2$ broaden the levels a , $M = \pm 1/2$ in the same way as do the transitions a , $M = +1/2 \rightleftharpoons a$, $M = -1/2$. However, the direct transitions $a \rightarrow b$ are considerably more probable than transitions accompanied by a reorientation of spin. The matrix element of the direct transitions calculated with the aid of (6) and (16) is in the first approximation:

$$\begin{aligned} \mathcal{H}_{a \rightarrow b}^{(2)} &= a(Q_5^2 - Q_6^2) + bQ_2Q_3 + cQ_5Q_6; \\ a &= \frac{9\sqrt{3}}{8}(\Lambda_1 - \frac{825}{8}\Lambda_2), \quad b = -18(\Lambda_1 - \frac{375}{16}\Lambda_2), \\ c &= i\frac{45\sqrt{3}}{8}\Lambda_2; \quad \Lambda_1 = ee'\alpha\bar{r}^2R^{-5}, \quad \Lambda_2 = ee'\rho\bar{r}^4R^{-7}, \\ \alpha &= -3\beta = 2/21. \end{aligned} \quad (17)$$

[The probabilities of the transitions $a \rightarrow c$, d , e , which are proportional to the Boltzmann factors $\exp(-\Delta/2kT)$, are small and are therefore not considered.] At the same time the matrix element for the transitions accompanied by a reorientation of spin a , $M = 1/2 \rightleftharpoons a$, $M = -1/2$ arises only in the third approximation taking into account the energies of the Zeeman and the spin-orbit interactions of the ion:

$$\mathcal{H}_{a,1/2 \rightarrow a,-1/2} \approx (\lambda g \beta H_0 / \Delta^2) \mathcal{H}_{ab}^{(2)} \approx 10^{-5} \mathcal{H}_{ab}^{(2)}. \quad (18)$$

We also note that the contribution to the width made by spin-relaxation transitions will be proportional to the square of the field H_0 , while the contribution of the direct transition $a \rightarrow b$ is independent of the field; experiments also did not show any apparent dependence of the width on the field.^[19]

We now evaluate the probability of the transition $a \rightarrow b$. The matrix element (17) is invariant under rotations of the complex ion; therefore, in writing down the probability of the transition $a \rightarrow b$ we should set $\tau_r = \infty$ in formulas (5). Moreover, the transition frequency $\omega_{ab} \approx 10^3 \text{ cm}^{-1}$ is large compared to λ_j , so that

$$\begin{aligned} \rho_{jj} &= 4\lambda_j/3\omega_{ab}^2 + 2\lambda_j/(\omega_{ab} - 2\omega_j)^2 + 2\lambda_j/(\omega_{ab} + 2\omega_j)^2, \\ \rho_{ij} &= \lambda_j(2/\omega_{ab}^2 + (\omega_{ab} - 2\omega_j)^{-2} + (\omega_{ab} + 2\omega_j)^{-2}). \end{aligned} \quad (19)$$

By utilizing (2) and (17) we obtain for the probability of transitions $a \rightarrow b$

$$\begin{aligned} w_{a,b} &= \hbar^{-2} [a^2(\bar{Q}_5^4 \rho_{55} + \bar{Q}_6^4 \rho_{66}) + b^2 \bar{Q}_2^2 \bar{Q}_3^2 \rho_{23} \\ &\quad + c^2 \bar{Q}_4^2 \bar{Q}_5^2 \rho_{45}] e^{-\delta/2kT}. \end{aligned} \quad (20)$$

The temperature dependence of (20) is given by the relation

$$w_{a,b} \sim \exp(-\delta/2kT) \text{cth}^2(\hbar\omega_j/2kT) \lambda_j, \quad (21)$$

with $\lambda_j \sim \exp(-E/RT)$ where E is a parameter which has the significance (and the order of magnitude) of the viscous barrier for the liquid.^[6] We set $\delta = 10^3 \text{ cm}^{-1}$, $\hbar\omega_j = 500 \text{ cm}^{-1}$, $E = 2 \text{ kcal/mole} = 700 \text{ cm}^{-1}$, then in accordance with (21) the probability $w_{a,b}$ increases by approximately a factor of 4.2 as the temperature of the liquid varies from 300 to 400° K. In accordance with Kozyrev's experiment,^[8] an increase in the line width by a factor of 1.8 is observed within the same temperature interval, and this is close to the calculated increase in width. The order of magnitude of (20) also agrees with the width observed in an aqueous solution. By setting, in agreement with Van Vleck,^[7] $R = 2A$, $\bar{r}^2 = 4.4 a_0^2$, $\bar{r}^4 = 31.2 a_0^4$, $e' = e$, we obtain $|a| = 2.4 \times 10^4$, $|b| = 6 \times 10^4$, $|c| = 10^3$; further, at $T = 350^\circ \text{ K}$ we have $\mu_f = 3 \times 10^{-23} \text{ g}$ (the mass of a molecule of water), $\omega_j = 500 \text{ cm}^{-1} = 10^{14} \text{ sec}^{-1}$ and $\bar{Q}^4 = 1.5 \times 10^{-37} \text{ cm}^4$; finally, on taking $\rho_{ij} \approx 2\rho_{ij} \approx 10^{-16}$ and $\exp(-\delta/2kT) \approx 0.1$, we obtain $w_{a,b} \approx 5 \times 10^9 \text{ sec}^{-1}$.

¹ V. B. Davenport and V. L. Root, Introduction to the Theory of Random Processes and Noise (Russian transl.) IIL, 1960.

² C. Kittel, Elementary Statistical Physics (Russian transl.) IIL, 1960.

³ J. R. Senitzky, Phys. Rev. **119**, 670 (1960).

⁴ S. A. Al'tshuler and K. A. Valiev, JETP **35**, 947 (1958), Soviet Phys. JETP **8**, 661 (1959).

⁵ I. V. Aleksandrov and G. M. Zhidomirov, JETP **41**, 127 (1961), Soviet Phys. JETP **14**, 94 (1961).

⁶ K. A. Valiev, JETP **40**, 1832 (1961), Soviet Phys. JETP **13**, 1287 (1961).

⁷ J. H. Van Vleck, Phys. Rev. **57**, 426 (1940).

⁸ L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Gostekhizdat, 1951.

⁹ Avvakumov, Garif'yanov, Kozyrev, and Tishkov, JETP **37**, 1564 (1959), Soviet Phys. JETP **10**, 1110 (1960).

¹⁰ H. J. McConnell, J. Chem. Phys. **25**, 709 (1956).

¹¹ B. R. McGarvey, J. Phys. Chem. **61**, 1232 (1957).

¹² B. Bleaney and K. W. H. Stevens, Rep. Progr. Phys. **16**, 108 (1953).

Translated by G. Volkoff
134

SCATTERING EQUATIONS AT LOW ENERGIES

V. V. MALAKHOV

Institute of Nuclear Physics, Moscow State University

Submitted to JETP editor February 8, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 762-764 (September, 1961)

Equations are proposed for the partial scattering amplitudes at low energies. The derivation of the equations is based on the double Mandelstam representation and the unitarity condition in the two-particle approximation. The characteristic feature of the equations is that the number of subtractions allowed in them is independent of the number of waves taken into account.

WE consider here the expansions of the scattering amplitude in partial-waves, used in the derivation of equations for the partial amplitudes in the low-energy region by methods based on the Mandelstam representation and unitarity condition in the two-particle approximation.^[1-3]

The results of this investigation enable us to obtain for the partial amplitudes at low energies new equations, in which the number of subtractions is independent of the number of waves taken into account.

We base our analysis on the scattering of scalar particles of unit mass. Let s_1, s_2 , and s_3 be the Mandelstam variables and let z_1, z_2 , and z_3 be the corresponding cosines of the scattering angles. We start from the double Mandelstam representation with one subtraction (see, for example,^[4]), which can be written in the form^[5]

$$A(s_1, s_2, s_3) = \lambda + \sum_{i=1}^3 \alpha(s_i, z_i);$$

$$\alpha(s_i, z_i) = (s_i - s_{i0}) \int_4^{\infty} \frac{a_i(x) dx}{(x - s_i)(x - s_{i0})} + (s_i - s_{i0}) \int_4^{\infty} \frac{dx}{(x - s_i)(x - s_{i0})}$$

$$\times \int_{16}^{\infty} dy \rho(x, y) \left[\frac{s_j - s_{j0}}{(y - s_j)(y - s_{j0})} + \frac{s_k - s_{k0}}{(y - s_k)(y - s_{k0})} \right],$$

(1)

where $i \neq j \neq k$.

The partial amplitudes $f_l(s)$ are the coefficients of the expansion of the scattering amplitude in Legendre polynomials of the cosines of the angles in reaction I:

$$A(s_1, s_2, s_3) = \sum_l \tilde{f}_l(s_1) P_l(z_1) \quad (2)$$

We introduce the quasi-partial amplitudes defined by the relation

$$\tilde{f}_l(s) = \int_{-1}^1 dz P_l(z) \alpha(s, z). \quad (3)$$

Unlike the partial amplitudes, the quasi-partial amplitudes are the coefficients of the expansion of only part of the scattering amplitude in terms of angles of one process.

Let us write down the formal expansion of the amplitude in quasi-partial waves; this expansion is known to be correct in the region of elastic scattering ($s_1, s_2, s_3 < 16$):

$$A(s_1, s_2, s_3) = \lambda + \sum_{i,l} \tilde{f}_l(s_i) P_l(z_i). \quad (4)$$

It follows from the crossing symmetry that we need sum only over even l .

From (1) and (3) we see that $\tilde{f}_l(s_1)$ has a left-hand cut in s_1 when $s_1 < -12$ (this cut is connected with the vanishing of the denominators in the integral with respect to y in formula (1), possible only when s_1 or s_3 are greater than 16).

It is easy to show from (3) and (4) that $\tilde{f}_l(s)$ has a zero of order l at the point $s = 4$. Taking into account the unitarity condition in the two-particle approximation, taking into account also the threshold behavior of $\tilde{f}_l(s)$ and neglecting the left-hand cut of $\tilde{f}_l(s)$, we can write for the quasi-partial amplitude a dispersion equation which together with (4) yields the following integral representation for the scattering amplitude:

$$A(s_1, s_2, s_3) = \lambda + \sum_{i,l} \tilde{f}_l(s_i) P_l(z_i); \quad (5)$$

$$\tilde{f}_l(s_i) = (s_i - s_{i0})(s_i - 4)^l \int_4^{\infty} \frac{|f_l(x)|^2 dx}{(x - s_i)(x - s_{i0})(x - 4)^l}.$$

For large values of x , the factor $\left(\frac{s_i - 4}{x - 4}\right)^l$ in (5)

tends to unity and consequently does not change the

asymptotic behavior of $\tilde{f}_l(s)$. This factor compensates for the pole in $P_l(z_j)$, since $z_i = (s_j - s_k)/(s_i - 4)$.

It is seen from (5) that neglect of the l -th partial wave in the physical region of the corresponding process is equivalent to neglecting the l -th quasi-partial wave in the physical regions of all the processes.

From representation (5) we can obtain equations for the partial amplitudes as proposed in several papers.^[1-3]

Investigating the analytic properties of the partial amplitudes with the aid of representation (5), we obtain the Chew and Mandelstam equations^[1] (this was done for $K\pi$ scattering in a recently published paper by Gourdin et al.^[6]). When $z_1 = 1$, the term $i = 2$ has the form

$$P_l\left(1 - \frac{s_1}{2}\right)(-s_{20})4^l \int_4^\infty \frac{|f_l(x)|^2 dx}{x(x-s_{20})(x-4)^l}. \quad (6)$$

This term increases at infinity as s_1 raised to the number of waves taken into account, and calls for the same number of subtractions in the equations of Chew and Mandelstam (more accurately, an analogous term in the absorptive part).

Essentially the same fact is noted in the paper of Efremov et al.^[2]. Choosing $s_{20} = 0$, we leave an analogous term in the derivative of the amplitude with respect to z_1 for $z_1 = 1$.

The equations of^[3] are obtained by neglecting the term with $i = 2$, which in the vicinity $4 < s < 16$ can be readily shown to be greater than the term with $i = 3$, which gives the left-hand cut in s_1 . It can be shown that the polynomial (6) cannot be compensated for in the region of elastic scattering by the analytic contribution of the higher approximations. Thus, the analytic continuation in the region $s_1 > 16$ for the forward scattering amplitude distorts the asymptotic properties of the solutions of the equations for the partial amplitudes.

We shall show now that we can obtain equations with asymptotic properties independent of the number of waves taken into consideration. For this purpose we equate the right-hand parts of (5) and (2) and their derivatives with respect to z_1 at $z_1 = 0$ (scattering by 90°). The first equation of the system has the form

$$\sum_l f_l(s) P_l(0) = \lambda + \sum_l P_l(0)(s - s_{10})(s - 4)^l \times \int_4^\infty \frac{|f_l(x)|^2 dx}{(x-s)(x-s_{10})(x-4)^l} + 2 \sum_l P_l\left(\frac{3s+4}{s+4}\right) \left(2 - \frac{s}{2} - s_{20}\right) \left(2 - \frac{s}{2} - 4\right)^l \int_4^\infty \frac{|f_l(x)|^2 dx}{(x-2+s/2)(x-s_{20})(x-4)^l}, \quad (7)$$

where we put for simplicity $s_{20} = s_{30}$.

The angle 90° is singled out by the fact that in this case we obtain an optimal distance from the regions of non-vanishing spectral functions; this is quite important because the cosines of the scattering angles of reactions II and III in formula (5) are nonphysical.

The procedure proposed is being applied at the present time to specific processes.

The author is grateful to V. Ya. Fainberg for numerous discussions and valuable advice.

¹G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

²Efremov, Meshcheryakov, Chou, and Shirokov, Preprint E-539, Joint Inst. Nuc. Res., 1960.

³Hsien, Ho, and Zoellner, JETP 39, 1668 (1960), Soviet Phys. JETP 12, 1165 (1961).

⁴K. A. Ter-Martirosyan, JETP 39, 827 (1960), Soviet Phys. JETP 12, 575 (1961).

⁵M. Cini and S. Fabini, Ann. Physics 9, 352 (1960).

⁶Gourdin, Noirot, and Salin, Nuovo cimento 18, 651 (1960).

Translated by J. G. Adashko

WIDTH AND SHAPE OF MÖSSBAUER LINES IN SOLID SOLUTIONS

M. A. KRIVOGLAZ

Institute of Metal Physics, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor February 23, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 765-772 (September, 1961)

The Mössbauer effect in solid solutions is studied. We take into account explicitly the dependence of the change in vibrational energy, which is associated with the change in mass of the emitting or absorbing nucleus, on the configuration of the atoms of different types surrounding this nucleus. An estimate is made of the resulting broadening of the Mössbauer line. The line shape in solid solutions is considered for the limiting cases of large and small natural width. Other causes of broadening of the Mössbauer line in solid solutions are mentioned.

THE position of the line from no-phonon absorption or emission of photons by nuclei in a crystal (the Mössbauer line^[1]) depends slightly on temperature.^[2,3] This dependence results from the small difference in masses of the nuclei in their ground and excited states, which leads to a difference in the vibrational energy E_n of the lattice in the initial and final states, even when the oscillator quantum numbers n are not changed. In ideal crystals the same energy difference E_n appears for excitation of any nucleus in the lattice so that, as shown by Snyder and Wick,^[4] there is only a shift of the line but no broadening. In solid solutions, however, this change in energy E_n must depend on the type of atom located in the neighborhood of the radiating (or absorbing) nucleus. Thus the position of the maximum of the Mössbauer line will be different for different configurations of the atoms in the solution surrounding the radiating nucleus, and the resulting spectral distribution should be broadened in accordance with the numerous different configurations around the radiating nucleus. In the present paper we treat the width and shape of such a spectral distribution.

Let us consider a binary substitutional solution A — B, in which photons can be emitted (absorbed) by the nuclei A. To simplify the formulas, we shall restrict ourselves to the case where the relative difference in masses of the component atoms is much greater than the difference in the interatomic force constants, and shall not take the latter effect into account. Then the problem reduces to finding the change δE_n in the vibrational energy when the mass of the s -th atom A changes by $\delta m = \Delta E/c_1^2$ (where ΔE is the difference in energy of the nucleus in the ground and excited

states, and c_1 is the velocity of light) as a function of the distribution of masses m_A and m_B on the other lattice sites s . We shall characterize the distribution of A and B atoms over the lattice sites by giving the numbers c_s , which are equal to 1 or 0, according as the site s is occupied by an A or by a B atom. The dependence of the vibration energy change on configuration is then determined by the dependence of δE_n on the variables c_s . This dependence can be found explicitly for the case of small mass difference of the atoms, or when the concentration of one of the components is low.

It is obvious that

$$\delta E_n = \frac{\partial E_n}{\partial m_{s'}} \delta m = \left(\psi_n^* \frac{\partial H}{\partial m_{s'}} \psi_n \right) \delta m = \frac{\hbar^2 \delta m}{2m_A^2} (\psi_n^* \Delta_{s'} \psi_n),$$

where H is the oscillation Hamiltonian and $\Delta_{s'}$ is the Laplacian for the atom s' . For sufficiently small $m_A - m_B$, no local oscillations appear in the crystal. According to^[4], we do not have to take account of fluctuations of the oscillator quantum numbers, and can replace these quantum numbers by their average values in the final expression for δE_n . Obviously we can then write δE_n in the form

$$\delta E_n = \frac{1}{2} \hbar^2 m_A^{-2} \delta m \text{Sp} [\Delta_{s'} e^{-\lambda H}] [\text{Sp} e^{-\lambda H}]^{-1}, \quad (1)$$

where $\lambda = 1/kT$, and the trace is taken over the complete system of oscillation wave functions for a given configuration of A and B atoms.

As the zeroth approximation for the problem of oscillation of a solid solution containing atoms of different masses we choose a crystal with the average reciprocal mass of the atoms $m^{-1} = m_s^{-1}$. We assume that there is one atom in the unit cell

of such a crystal. The Hamiltonian for the vibrations of the solid solution is expressed in terms of the phonon creation and annihilation operators $a_{\mathbf{k}j}^\dagger, a_{\mathbf{k}j}$ (where $\mathbf{k}$ is the wave vector and j the branch number) as follows (cf., for example, [5]):

$$H = \frac{1}{2} \sum_{\mathbf{k}j} \hbar \omega_{\mathbf{k}j} (a_{\mathbf{k}j}^\dagger a_{\mathbf{k}j} + a_{\mathbf{k}j} a_{\mathbf{k}j}^\dagger) + H', \quad (2)$$

$$H' = \frac{1}{2} \sum_{\mathbf{k}j\mathbf{k}'j'} V_{\mathbf{k}j\mathbf{k}'j'} a_{\mathbf{k}j} a_{\mathbf{k}'j'}^\dagger + \frac{1}{2} \sum_{\mathbf{k}j\mathbf{k}'j'} V'_{\mathbf{k}j\mathbf{k}'j'} a_{\mathbf{k}j} a_{\mathbf{k}'j'} + \text{H.c.}$$

Here

$$V_{\mathbf{k}j\mathbf{k}'j'} = \frac{\hbar}{2N} \frac{(m_B - m_A)m}{m_A m_B} \sqrt{\omega_{\mathbf{k}j} \omega_{\mathbf{k}'j'}} (\mathbf{e}_{\mathbf{k}j} \mathbf{e}_{\mathbf{k}'j'}^*) \sum_{\mathbf{s}} (c_s - c) \times \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}_s], \quad (3)$$

N is the number of atoms, $\mathbf{e}_{\mathbf{k}j}^* = -\mathbf{e}_{-\mathbf{k}j}^*$ are polarization unit vectors, c is the concentration of A atoms. The expression for $V'_{\mathbf{k}j\mathbf{k}'j'}$ is obtained from (3) by replacing $\mathbf{k}'$ by $-\mathbf{k}'$. Similarly, the operator $\hbar^2 \delta m (2m_A^2)^{-1} \Delta_{\mathbf{s}'}$ is expressed in terms of $a_{\mathbf{k}j}^\dagger, a_{\mathbf{k}j}$ by formula (2) for H' , if we replace

$$(m_A - m_B) \sum_{\mathbf{s}} (c_s - c) \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}_s] \text{ by } \frac{m_B}{m_A} \delta m \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}_{s'}]$$

in expression (3) for $V_{\mathbf{k}j\mathbf{k}'j'}$.

Substituting the expressions for H and $\Delta_{\mathbf{s}'}$ in (1), expanding the exponential operator in series in H' and stopping with the linear terms, we can easily compute the trace in (1). As a result we find that, to terms of order $\delta m (m_A - m_B)$ inclusive, δE_n can be represented as a linear form in the c_s :

$$\delta E_n = \hbar \alpha^0 + \sum_{s \neq 0} \hbar \alpha_s (c_s - c). \quad (4)$$

Here

$$\alpha_s = \frac{1}{4N^2} \frac{m_A - m_B}{m} \frac{\delta m}{m} \sum_{\mathbf{k}j\mathbf{k}'j'} \omega_{\mathbf{k}j} \omega_{\mathbf{k}'j'} |\mathbf{e}_{\mathbf{k}j} \mathbf{e}_{\mathbf{k}'j'}^*|^2 \cos(\mathbf{k} - \mathbf{k}', \mathbf{R}_s) \times \left[\frac{n_{\mathbf{k}j} - n_{\mathbf{k}'j'}}{\omega_{\mathbf{k}j} - \omega_{\mathbf{k}'j'}} - \frac{1 + n_{\mathbf{k}j} + n_{\mathbf{k}'j'}}{\omega_{\mathbf{k}j} + \omega_{\mathbf{k}'j'}} \right], \quad (5)$$

$$\alpha^0 = -\frac{m \delta m}{2m_A^2 N} \sum_{\mathbf{k}j} \omega_{\mathbf{k}j} \left(n_{\mathbf{k}j} + \frac{1}{2} \right) + \alpha_0,$$

where the $n_{\mathbf{k}j}$ are the average quantum numbers of the oscillators $\mathbf{k}j$, α_0 is the value of α_s for $\mathbf{R}_s = 0$, and the coordinate origin is taken at the site $\mathbf{s}'$.

If the mass difference of the atoms is large, but the concentration of one of the components is so small that we can neglect configurations in which there are two (or more) impurity atoms in the neighborhood of the given site, then δE_n

can again be written as the linear expression (4) in the c_s . The quantities α^0, α_s appearing in this expression are, however, not given by formulas (5) in this case, but rather by more complicated formulas, which can be found without the use of perturbation theory by the methods which have been applied in numerous papers. [6]

The dependence of the change in vibrational energy δE_n on the atom configuration (on the c_s) results in a broadening of the Mössbauer line. Obviously the intensity distribution will be qualitatively different in those cases where this broadening is much larger or much smaller than the natural width Γ . First let us consider the case where the broadening is much greater than Γ , and set $\Gamma = 0$. Then the spectral distribution of the intensity of the no-phonon emission of photons with wave vector $\mathbf{q}$ by the nucleus $\mathbf{s}'$, per unit solid angle and per unit frequency range, is given by the formula

$$\sigma(\mathbf{q}, \omega) = \frac{\sigma^0}{4\pi c} \langle c_{s'} D \delta(\omega - \delta E_n / \hbar) \rangle = \frac{\sigma^0}{4\pi c} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle c_{s'} D \exp(-i\delta E_n t / \hbar) \rangle, \quad (6)$$

$$D = \exp[-M_{s'}(c_s, \mathbf{q})].$$

Here σ^0 is the integral intensity of emission in the absence of vibration, D is the Debye factor for attenuation of the intensity of the no-phonon emission for the site $\mathbf{s}'$, ω is measured from the frequency $\Delta E / \hbar$, corresponding to the unshifted emission line, $\langle \dots \rangle$ denotes an average over all possible configurations of atoms on the lattice sites. Like δE_n , D also depends on the configuration of the atoms around the site $\mathbf{s}'$. But the differences in D for different configurations are proportional to $m_A - m_B$. Thus in determining the spectral distribution, neglecting terms $\sim (m_A - m_B)^2$, we can take D with the average value $\exp[-M_A(\mathbf{q})]$.

The spectral shape is conveniently characterized by giving the moments of the function $\sigma(\mathbf{q}, \omega)$. Substituting (4) in (6) and integrating by parts, we find that the moments are given by the formulas

$$\sigma_m(\mathbf{q}) = \int (\omega - \alpha^0)^m \sigma(\mathbf{q}, \omega) d\omega; \quad \sigma_0(\mathbf{q}) = \int \sigma(\mathbf{q}, \omega) d\omega = \frac{\sigma^0}{4\pi} \exp[-M_A(\mathbf{q})]; \quad \sigma'_m = \frac{\sigma_m(\mathbf{q})}{\sigma_0(\mathbf{q})} = \frac{1}{c} \langle c_{s'} \left[\sum_{s \neq 0} \alpha_s (c_s - c) \right]^m \rangle \quad (7)$$

(the site $\mathbf{s}'$ is at the origin of coordinates).

Formulas (7) express the moments in terms of the constants α_s , the concentration, and the correlation parameters. In the case of an ideal solution, where there is no correlation, $\langle c_s^m \rangle = c$,

$\langle c_{S_1}^{m_1} c_{S_2}^{m_2} \rangle = c^2 (s_1 \neq s_2)$ etc, and it is easy to obtain expressions which give explicitly the concentration dependence of the first few moments:

$$\begin{aligned} \sigma'_1 &= 0, \quad \sigma'_2 = c(1-c) \sum_{s \neq 0} \alpha_s^2, \quad \sigma'_3 = c(1-c)(1-2c) \sum_{s \neq 0} \alpha_s^3, \\ \sigma'_4 &= c(1-c)(1-3c+3c^2) \sum_{s \neq 0} \alpha_s^4 + c^2(1-c)^2 \sum_{s_1, 2 \neq 0, s_1 \neq s_2} \alpha_{s_1}^2 \alpha_{s_2}^2. \end{aligned} \quad (8)$$

The first moment σ'_1 determines the average frequency of the emitted radiation. Since $\sigma'_1 = 0$, in ideal solutions the shift of the average frequency resulting from the change δm in the mass of the radiating nucleus is determined by the quantity α^0 . Except for a numerical factor which is almost equal to one, the value of $\sqrt{\sigma'_2}$ is equal to the half-width δ of the spectral curve. It follows from (8) that, in the case we are considering, the concentration dependence of the half-width of the curve is determined by the factor $\sqrt{c(1-c)}$. At high temperatures, according to (5) and (8), the half-width is proportional to T , and as $T \rightarrow 0$ it tends to a nonzero limit. The third moment σ'_3 determines the asymmetry of the spectral distribution. In ideal solutions with stoichiometric composition ($c = 1/2$), the asymmetry disappears, but for other concentrations the asymmetry index is different from zero and must have opposite sign for $c > 1/2$ and $c < 1/2$. The spectral curve is then obviously significantly different from the Lorentz shape.

For a rough estimate of the order of magnitude of the broadening δ , we consider, for example, solutions with a face-centered cubic lattice, and include in the sum over s for σ'_2 only the terms corresponding to the first and second coordination spheres (for large R_s , α_s falls off like R_s^{-7}). For $T = 0$, in the Debye approximation, it follows from (5) that

$$\alpha_1 \ll \alpha_2, \quad \alpha_2 \sim -\frac{1}{3} \cdot 10^{-2} \omega_m \frac{\Delta m \delta m}{m} \sim -\frac{2}{3} \cdot 10^{-2} \alpha^0 \frac{\Delta m}{m}$$

($\Delta m = m_A - m_B$, ω_m is the maximum vibration frequency, α^0 is the line shift). From this it follows that δ is of order

$$\delta \sim 2\sqrt{z_2} \sqrt{c(1-c)} \alpha_2 \sim 0.03 \sqrt{c(1-c)} \alpha^0 \Delta m / m$$

(where z_2 is the coordination number). Since the value of α^0 varies between the limits $\hbar \alpha^0 \sim 10^{-9} - 3 \times 10^{-8}$ ev, in concentrated solutions ($c \sim 1/2$) with large relative mass difference of the atoms ($|\Delta m| \sim m$) the width can be of order $\delta \sim 10^{-10} - 10^{-9}$ ev. Thus δ can be much greater than the natural width Γ only for the narrowest lines, such as for example Zn^{67} ($\Gamma = 5 \times 10^{-11}$ ev). We should note that the use of the true vibration spectrum in place of the Debye spectrum in evaluating the sums (5) may change this estimate.

In the case of nonideal solutions, the moments depend not only on the concentration but also on the parameters determining the degree of order in the solution. For example, when there is short range order in disordered solutions, the first moment, according to (7), is given by the formula

$$\sigma_1 = \sum_{s \neq 0} (p_s - c) \alpha_s, \quad (9)$$

where p_s is the a posteriori probability of finding an A atom at site s if there is an A atom at the origin. Thus when short range order is established in a solution one may observe a significant shift in the Mössbauer line. Similarly, short range order may affect the width and shape of the line. We also see from (7) that the second and higher moments depend on the parameters for short range order for three and more sites.

The establishment of long range order may have an even more marked influence on the Mössbauer effect. To be specific, let us consider solutions with a body-centered cubic lattice, like β -brass (CsCl type). When long range order is present, the probabilities of occupying sites of the first and second types with A atoms are different: $p_A^{(1)} = c + \eta/2$ and $p_A^{(2)} = c - \eta/2$ (where η is the degree of long range order). In the case of an ordered solution, the energy change δE_n given by (4) is conveniently written as

$$\delta E_n = \hbar \alpha^0 + \hbar \alpha^\pm + \sum_{s \neq 0} \hbar \alpha_s (c_s - \bar{c}_s).$$

Here $\bar{c}_s$ is equal to $p_A^{(1)}$ or to $p_A^{(2)}$, depending on the type of the site s ;

$$\alpha^\pm = \pm \frac{1}{2} \eta \left(\sum_{s \neq 0}'' \alpha_s - \sum_s' \alpha_s \right),$$

where the plus sign corresponds to a radiating nucleus on a site of the first type, and the minus sign to one on a site of the second type; the summation Σ'' goes over even coordination spheres, the sum Σ' over odd ones.

For the case of large values of η , the correlation in the solution is unimportant. Neglecting correlations, we can easily determine the moments for the spectral curves corresponding to radiation by nuclei on sites of the first and second types, in the same way as we did above, and then we can find the moments for the resulting intensity distribution by combining the results. The first moment is equal to

$$\sigma'_1 = \frac{\eta^2}{4c} \left(\sum_{s \neq 0}'' \alpha_s - \sum_s' \alpha_s \right). \quad (10)$$

Thus the shift of the average frequency ω relative to α^0 when long range order is established, is proportional to η^2 . The second moment, defined

relative to the average frequency $\alpha_0 + \alpha'_1$ (the dispersion of the curve) is equal to

$$\begin{aligned} \sigma'_2 &= \frac{1}{\sigma_0(q)} \int (\omega - \alpha^0 - \alpha'_1)^2 \sigma(q, \omega) d\omega \\ &= \frac{1-c}{c} p_A^{(1)} p_A^{(2)} \sum_s \alpha_s^2 + \frac{1}{2c} [(p_A^{(1)})^2 (1 - p_A^{(1)}) \\ &\quad + (p_A^{(2)})^2 (1 - p_A^{(2)})] \sum_{s \neq 0} \alpha_s^2 \\ &\quad + \frac{\eta^2}{4c^2} p_A^{(1)} p_A^{(2)} \left(\sum_s \alpha_s - \sum_{s \neq 0} \alpha_s \right)^2. \end{aligned} \quad (11)$$

For solutions with stoichiometric composition ($c = 1/2$), this formula simplifies:

$$\sigma'_2 = \frac{1}{4} (1 - \eta^2) \sum_{s \neq 0} \alpha_s^2 + \frac{1}{4} \eta^2 (1 - \eta^2) \left(\sum_s \alpha_s - \sum_{s \neq 0} \alpha_s \right)^2. \quad (12)$$

As it should be, with ordering of the solution, when $\eta \rightarrow 1$, the dispersion and width of the curve tend to zero.

Let us now take into account the finite value of the natural width Γ of the emission line. Expression (4) then determines the positions of the maxima of the Lorentz curves with width Γ for the different atom configurations in the solution. For nonzero, but small, Γ ($\Gamma \ll \delta$) it is obvious that the spectral curve in the region of the maximum $|\omega - \alpha^0| \lesssim \delta$ is only slightly smeared out. However, including the finite value of Γ essentially changes the spectral distribution in the "skirts" of the curve, for $|\omega - \alpha^0| \gg \delta$. Whereas, when $\Gamma = 0$, the quantity $\sigma(q, \omega)$ falls off faster than any power of $1/\omega$ (all the moments σ_m are finite), for $\Gamma \neq 0$, in the region $|\omega - \alpha^0| \gg \delta$ the quantity $\sigma(q, \omega) \approx \sigma_0 \Gamma |2\pi(\omega - \alpha^0)^2|^{-1}$ falls off like $(\omega - \alpha^0)^{-2}$.

It is easy to see that including the finite width Γ in the integral representation (6) for $\sigma(q, \omega)$ reduces to adding the factor $\exp(-\Gamma|t|/2)$. In particular, for ideal solutions, performing the average, we get

$$\begin{aligned} \sigma(q, \omega) &= \sigma_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp \left\{ i \left(\omega - \alpha^0 + c \sum_{s \neq 0} \alpha_s \right) t \right\} \\ &\quad \times \prod_{s \neq 0} [1 - c + ce^{-i\alpha_s t}] \exp(-\Gamma|t|/2). \end{aligned} \quad (13)$$

Performing the integration, we can represent $\sigma(q, \omega)$ as a sum of Lorentz curves:

$$\begin{aligned} \sigma(q, \omega) &= \frac{\sigma_0}{2\pi} \sum_{\dots p_i \dots} A_p \Gamma \left\{ \left[\omega - \alpha^0 \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^{\infty} \alpha_i (p_i - cz_i) \right]^2 + \Gamma^2/4 \right\}^{-1}. \end{aligned} \quad (14)$$

Here i is the number of the coordination sphere; α_i is the value of α_s for the i -th sphere; $p_i = 1, 2, \dots$, z_i, z_i are the coordination numbers;

$$A_p = \prod_{i=1}^{\infty} \frac{z_i!}{p_i! (z_i - p_i)!} c^{p_i} (1 - c)^{z_i - p_i}.$$

The superposition of Lorentz curves (14) usually describes a smooth spectral distribution. Only in special cases, when Γ is very small and the concentration of one of the components is low (or the solution is almost completely ordered) can one observe the fine structure of this distribution, which is given for ideal solutions by formula (14). The shape of the smooth curve which is obtained for $\sigma(q, \omega)$ when Γ is small ($\Gamma \ll \delta$) was discussed above. Now we shall investigate the opposite limiting case, when $\Gamma \gg \sqrt{\sigma'_2}$, i.e., the broadening caused by the dependence of δE_n on atomic configuration is much smaller than the natural width. In this case we can make an expansion in (14) (or in the analogous expression for nonideal solutions) in powers of α and keep only the leading terms in the expansion. Then the characteristics of the spectrum can be expressed in terms of the same constants as for the case of $\Gamma \ll \delta$, if we note that

$$\sum_{\dots p_i \dots} A_p \left[\sum_i (p_i - cz_i) \alpha_i \right]^m = \sigma'_m.$$

As a result we find that the frequency ω_m corresponding to the maximum of the spectral curve is given by the following formula, which is valid for both nonideal and ideal solutions:

$$\omega_m = \alpha^0 + \alpha'_1 - 8\sigma'_2/\Gamma^2. \quad (15)$$

Thus the dependence of ω_m on the order parameters for large values of Γ is found to be the same (neglecting higher order terms) as that found above for the average frequency when Γ is small.

We get different corrections ϵ to the width $\Gamma + \epsilon$ of the spectral curve for $\sigma'_1 = 0$ if we define it as the integral width or as the width at half maximum. This correction is expressible in terms of the second moment, and is equal to

$$\epsilon = 4\sigma'_2/\Gamma, \quad (16)$$

in the first case, and to $\epsilon = 6\sigma'_2/\Gamma$ in the second. From (8) and (16) we see that for large Γ the width depends linearly on $c(1-c)$ and not on $\sqrt{c(1-c)}$, as it does for small Γ . In contrast to the situation for small Γ , where this type of broadening is proportional to $(m_A - m_B)/m$, for large Γ the correction ϵ is proportional to $(m_A - m_B)^2/m^2$, and is unimportant when the relative mass difference is small. The dependence of the maximum intensity $\sigma_m(q, \omega)$ on the constants α is determined by the factor $(1 - \epsilon/\Gamma)$.

The asymmetry index of the curve is expressed in terms of the third moment, and is equal to

$$(\omega_{+1/2} - \omega_{-1/2}) / \Gamma \approx 8\sigma'_3 / \Gamma^3. \quad (17)$$

As for the case of small Γ , the asymmetry vanishes in the ideal solution when $c = 1/2$. At the same time the asymmetry index in this case is $\sim (\Gamma/\delta)^3$ times smaller than for small Γ .

As already mentioned, the integral intensity of the line is determined by the average Debye factor $\langle D \rangle = \exp[-M_A(q)]$. We know that in ideal cubic crystals, in the harmonic approximation, M_A is proportional to q^2 . But in solid solutions, according to [7], $M_A \sim q^2$ only for small q . For large q the Debye factor is a more complicated function of q . In particular this has the consequence that even in cubic crystals M_A depends not only on the magnitude but also on the direction of the vector q . Thus if the nuclei emit high energy photons (large q) one should observe a dependence of the emission (or absorption) intensity on the orientation of the emission direction relative to the axes of a cubic single crystal.

It has already been pointed out (cf., for example, [8]) that the intensity of the recoilless radiation can be increased by increasing the Debye factor, by introducing the radiating atom as an impurity in a crystal consisting of atoms with large masses. We have given an estimate of this effect. [7] Here we mention only that at high temperatures, when the classical approximation is applicable, M_A does not depend at all on the masses of the atoms of the solution, and M_A can be reduced only by increasing the atomic interaction constants.

When we go over from ideal crystals to solid solutions, the distribution in intensity of the one-phonon and multiphonon emission also changes. In particular, the singularities in the distribution function of the vibration frequencies and the singularities in the frequency dependence of the probability for one-phonon emission, which must occur in ideal crystals, are smeared out in solid solutions (cf. [5]).

We have here treated the broadening of the Mössbauer line associated with the dependence of

the change in vibrational energy δE_n on the atomic configuration. In a solid solution the difference in energy between the excited and ground states of the nucleus may also depend on the atomic configuration for other reasons, for example as a result of nuclear isomeric shift or magnetic shift of nuclear levels in a ferromagnet. In many cases these level shifts, in some approximation, depend additively on the nature of the surrounding atoms, i.e., they depend linearly on the numbers c_s in a formula of the type of (4). It is obvious that in these cases the spectral distribution of the Mössbauer line is again given by the above formulas. The only difference is that now the constants α_s which appear in these formulas have a different physical significance.

¹R. L. Mössbauer, *Z. Physik* **151**, 124 (1958); *Naturwiss.* **45**, 538 (1958); *Z. Naturforsch.* **14a**, 211 (1959).

²R. V. Pound and G. A. Rebka, *Phys. Rev. Letters* **4**, 274 (1960).

³B. D. Josephson, *Phys. Rev. Letters* **4**, 341 (1960).

⁴H. S. Snyder and G. C. Wick, *Phys. Rev.* **120**, 128 (1960).

⁵M. A. Krivoglaz, *JETP* **40**, 567 (1961), *Soviet Phys. JETP* **13**, 397 (1961).

⁶I. M. Lifshitz, *JETP* **17**, 1017, 1076 (1947); *Nuovo cimento, Suppl.* **3**, 716 (1956). M. Lax, *Phys. Rev.* **94**, 1391 (1954). G. F. Koster and J. C. Slater, *Phys. Rev.* **94**, 1392 (1954); **95**, 1167 (1954). E. W. Montroll and P. B. Potts, *Phys. Rev.* **100**, 525 (1955); **102**, 72 (1956).

⁷M. A. Krivoglaz, *Kristallografiya* **4**, 813 (1959), *Soviet Physics-Crystallography* **4**, 775 (1960); M. A. Krivoglaz and E. A. Tikhonova, *FMM (Physics of Metals and Metallography)*, in press.

⁸F. L. Shapiro, *Usp. Fiz. Nauk* **72**, 685 (1960), *Soviet Phys.-Uspekhi* **3**, 881 (1961).

DETERMINATION OF THE $K^+ \rightarrow 2\pi + \gamma$ DECAY AMPLITUDE FROM DISPERSION RELATIONS AND UNITARITY

I. G. IVANTER

Submitted to JETP editor March 3, 1961; resubmitted May 16, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 773-783 (September, 1961)

The effect of $\pi\pi$ interaction on the amplitude for the radiative θ^+ decay is taken into account. An expression for the amplitude is obtained in terms of P-wave $\pi\pi$ -scattering phase shifts. The Omnes-Muskhelishvili method for solving the integral equation is generalized to the case when the inhomogeneous part of the equation is a singular function.

THE radiative θ^+ decay $K^+ \rightarrow \pi^+ + \pi^0 + \gamma$ has been discussed by Good,^[1] without taking into consideration the final state interaction between the pions. In the present paper the interaction between the π mesons is taken into account and the πK interaction is neglected.

1. GENERAL FORM OF THE AMPLITUDE

The amplitude for the indicated process consists of two parts, corresponding to electric and magnetic radiations:

$$M = (k_\mu w_\nu - k_\nu w_\mu) p_\mu q_\nu \varphi + i \varepsilon_{\mu\nu\alpha\beta} (k_\alpha w_\beta - k_\beta w_\alpha) p_\mu q_\nu \chi. \quad (1)$$

Here k and w are respectively the 4-momentum and polarization of the photon; p and q are the 4-momenta of the π^+ and π^0 mesons; and φ and χ are functions of the invariants in the problem.

Internal bremsstrahlung of the K^+ and π^+ mesons contributes only to the "electric" part of the amplitude, i.e., to the function φ . The internal bremsstrahlung diagram is shown in Fig. 1. The corresponding amplitude is given by

$$M_T = \left(\frac{(Qw)}{(Qk)} - \frac{(pw)}{(pk)} \right) emG, \quad (2)$$

where G is the $K_{2\pi}^+$ decay coupling constant, m is the mass and Q is the 4-momentum of the K^+ meson. M_T may be expressed in the form

$$M_T = q_\mu p_\nu (k_\mu w_\nu - w_\mu k_\nu) \varphi_0; \quad (3)$$

$$\varphi_0 = emG / (pk) (Qk) = 4meG / (s - m^2) (t - \mu^2), \quad (4)$$

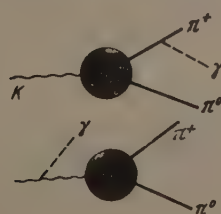


FIG. 1

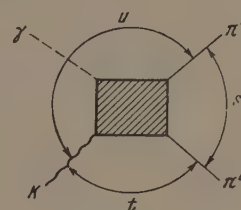


FIG. 2

where μ is the mass of the π^+ meson, and s and t are the invariants corresponding to the $\pi^+\pi^0$ and $\pi^0 K^+$ channels (see Fig. 2):

$$s = (p + q)^2, \quad t = (k + p)^2. \quad (5)$$

In addition to internal bremsstrahlung one has radiation from the block of heavy particles ("vertex" radiation). This radiation has an electric and a magnetic part. The part of the amplitude corresponding to this radiation depends only weakly on the invariants and may therefore be written in the following form

$$M_B = A p_\mu q_\nu (k_\mu w_\nu - k_\nu w_\mu) + D p_\mu q_\nu i \varepsilon_{\mu\nu\alpha\beta} (k_\alpha w_\beta - k_\beta w_\alpha). \quad (6)$$

Here A and D are constants connected with the electric and magnetic parts.

The electric part is of greatest interest because of the importance of the interference between the "vertex" radiation and the pole term. On the other hand the interference between the magnetic and electric radiations vanishes after summation over polarizations.^[1] Thus, what is taken into account below are chains of the kind illustrated in Fig. 3, as well as chains involving internal bremsstrahlung of the kind illustrated in Fig. 4.* We leave out contributions arising from radiation from the

*Chains with an odd number of pions give contributions that are even less than those that are referred to below as small, because virtual decays of K^+ mesons into an even number of pions are forbidden by isotopic spin selection rules.

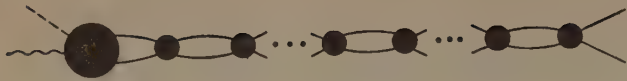


FIG. 3

$\pi\pi$ -interaction vertex, i.e., radiation from the entire block at the strong interaction vertex (see Fig. 5). The contribution from such diagrams is however less than the contribution from the diagrams shown in Fig. 3, because in the diagrams of Fig. 5 the $K_{2\pi}^+$ decay has to take place first and it is forbidden by the $\Delta T = 1/2$ rule. As noted by Good,^[1] this forbiddenness is lifted for that part of the amplitude which is connected with radiation from the heavy particles block in the "decay" vertex. As a consequence the ratio of the contributions from diagrams of Fig. 5 and Fig. 3 should be proportional to the ratio of the $K_{2\pi}^0$ and $K_{2\pi}^+$ decay constants, i.e., $\sim 1/20$. We ignore any effects due to the $K\pi$ interaction; some estimates are made in this connection at the end of the paper.

The calculation of the magnetic part of the amplitude is made difficult by the existence of diagrams shown in Fig. 6. Here the $K_{3\pi}$ decay takes place at the weak vertex, which is isotopically allowed; these diagrams are difficult to take into account.

2. SPECTRAL DECOMPOSITION AND UNITARITY CONDITION FOR THE ELECTRIC PART OF THE AMPLITUDE

The spectral decomposition for the electric part of the amplitude may be written in the form

$$\begin{aligned} \varphi(s, t) = & \varphi_0(s, t) + A + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \rho_1(s') \left(\frac{1}{s' - s - i\epsilon} - \frac{1}{s'} \right) ds' \\ & + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \rho_2(t') \left(\frac{1}{t' - t - i\epsilon} - \frac{1}{t'} \right) dt' \\ & + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \rho_3(u') \left(\frac{1}{u' - u - i\epsilon} - \frac{1}{u'} \right) du'. \end{aligned} \quad (7)$$

These dispersion relations have been written with a subtraction. The subtraction constant A is con-

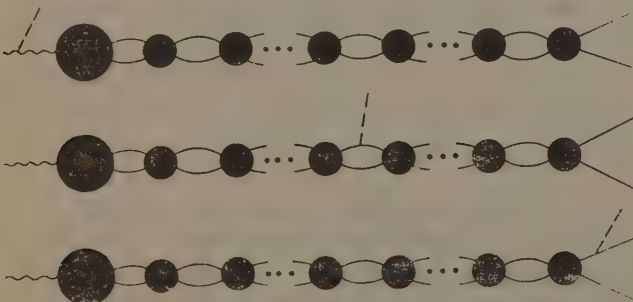


FIG. 4



FIG. 5

nected with radiation from the block of heavy particles.

The branch points in the invariants t and u lie significantly further than $4\mu^2$. This makes it possible to ignore the last two integrals in Eq. (7) (μ/m approximation). An estimate of these terms will be given below. (We may note here that the magnetic part of the amplitude cannot be calculated precisely because the branch point in the t -channel lies at also $4\mu^2$.) Thus we obtain, in first approximation, the following expression for the electric part of the amplitude

$$\varphi(s, t) = \varphi_0(s, t) + A + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \rho_1(s') \left[\frac{1}{s' - s - i\epsilon} - \frac{1}{s'} \right] ds'. \quad (8)$$

In writing Eq. (7) we have assumed that the channel with four pions, which gives rise to a branch point at $s = 16\mu^2$, can be neglected. If this channel cannot be neglected than instead of Eq. (8) we get the more general expression

$$\varphi(s, t) = \varphi_0(s, t) + A + \frac{1}{\pi} \iint \frac{\rho(s, t) st dt' ds'}{s' t' (s' - s) (t' - t)}. \quad (9)$$

By averaging Eqs. (4) and (7) over t for a fixed s we get

$$\varphi_0(s) = \frac{-4mGe\sqrt{s}}{(s-m^2)^2 \sqrt{s-4\mu^2}} \ln \left(\frac{\sqrt{s} + \sqrt{s-4\mu^2}}{2\mu} \right)^2;$$

$$\varphi_1(s) = A + \varphi_0(s) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \rho_1(s') \left[\frac{1}{s' - s - i\epsilon} - \frac{1}{s'} \right] ds'. \quad (10)$$

The unitarity conditions connect A and φ_0 with each other, but do not couple the electric and magnetic parts of the amplitude. The unitarity conditions give

$$\begin{aligned} \rho_1(s) p_\mu q_\nu (k_\mu w_\nu - k_\nu w_\mu) &= (k_\mu w_\nu - k_\nu w_\mu) \sqrt{1 - 4\mu^2/s} \\ &\times \frac{1}{4\pi} \int \varphi(s) p'_\mu q'_\nu \lambda_{\pi\pi} d\Omega \\ &= (k_\mu w_\nu - k_\nu w_\mu) p_\mu q_\nu e^{-i\delta_1} \sin \delta_1 \varphi(s), \end{aligned} \quad (11)$$

where δ_1 is the P -wave $\pi\pi$ scattering phase shift. Here we took into account the $\pi\pi$ scattering channel (see Fig. 3). If also the second and third channels ($K\pi$ scattering) are taken into account then the unitarity relations for φ become

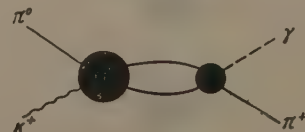


FIG. 6

$$\rho_1 = e^{-i\delta_1} \sin \delta_1 \varphi(s); \quad (12)$$

$$\rho_2 = e^{-i\delta'_1} \sin \delta'_1 \varphi(t), \quad \rho_3 = e^{-i\delta'_1} \sin \delta'_1 \varphi(u). \quad (13)$$

Here δ'_1 is the P-wave $K\pi$ scattering phase shift.

In first approximation we obtain according to Eq. (10) the following integral equation for the function $\varphi_1(s)$

$$\varphi_1(s) = \varphi_0(s) + A + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \left[\frac{1}{s' - s - i\varepsilon} - \frac{1}{s'} \right] e^{-i\delta_1} \sin \delta_1 \varphi_1 ds'. \quad (14)$$

3. SOLUTION OF INTEGRAL EQUATION

In order to solve Eq. (14) it is sufficient to solve an equation of the following type

$$\varphi(s) = \frac{q(s)}{[s - v - i\varepsilon]^n} + \frac{1}{\pi} \int_a^{\infty} \frac{h^*(s') \varphi(s')}{s' - s - i\varepsilon} ds', \quad (15)$$

$$h^* = e^{-i\delta_1} \sin \delta_1.$$

The function $q(s)$ appearing in Eq. (15) is regular in the region $a \leq s < \infty$. For $n = 0$ this type of equation has been solved by Omnes^[2] using Muskhelishvili's method. In generalizing the Omnes-Muskhelishvili method to the solution of Eq. (15) with $n \neq 0$ we found it necessary to consider integrals of the type

$$\int_a^{\infty} F(s') [(s' - v - i\varepsilon)^n (s' - s - i\varepsilon)]^{-1} ds', \quad (16)$$

where $F(s')$ is a regular function.

Let us evaluate first the integral

$$J = \int_L F(s') (s' - v)^{-n} ds' \quad (17)$$

where the contour L is shown in Fig. 7 and the function $F(s)$ is assumed to have the property that $\text{Im } F(s) = 0$ if $\text{Im } s = 0$. The imaginary part of this integral is given by

$$\text{Im } J = \pi \text{res } J = \pi F^{(n-1)}(v) [(n-1)!]^{-1}. \quad (18)$$

If $n = 1$ then

$$\text{Re } J = P \int_a^{\infty} F(s') (s' - v)^{-1} ds'.$$

However already for $n = 2$ the principal value integral

$$P \int_a^{\infty} F(s') (s' - v)^{-n} ds'$$



FIG. 7

diverges (equals $+\infty$). The real part of the integral J may be calculated in this case directly by integration along the contour; on the other hand it is equal to the generalized principal value integral, which we denote by PP, defined by

$$\text{PP} \int_a^{\infty} F(s') (s' - v)^{-n} ds' = \lim_{\varepsilon \rightarrow 0} P \int_a^{\infty} F(s') \left[\prod_{k=1}^n (s' - v - k\varepsilon) \right]^{-1} ds' = \text{Re } J. \quad (19)$$

Carrying out the integration around the pole we find

$$\text{PP} \int_a^{2v-a} \frac{ds'}{(s' - v)^n} = \frac{(-1)^{n-1} - 1}{(n-1)(v-a)^{n-1}}; \quad n > 1, \quad a < v. \quad (20)$$

Expanding the function $F(s)$ in a power series and making use of Eq. (20) we get

$$\begin{aligned} J &= \int_L \frac{F(s') ds'}{(s' - v)^n} = \int_a^{\infty} \frac{F(s') ds'}{(s' - v - i\varepsilon)^n} \\ &= i\pi \frac{F^{(n-1)}(v)}{(n-1)!} + \sum_{k=0}^{n-2} F^{(k)}(v) \frac{(-1)^{n-k-1} - 1}{(n-k-1)k!(v-a)^{n-k-1}} \\ &\quad + P \int_a^{2v-a} \left[F(s') - \sum_{k=0}^{n-2} \frac{(s-v)^k}{k!} F^{(k)}(v) \right] \frac{ds'}{(s' - v)^n} \\ &\quad + \int_{2v-a}^{\infty} F(s') \frac{ds'}{(s' - v)^n}. \end{aligned} \quad (21)$$

For integrals of the type

$$J_{1,2} = \int_{L_{1,2}} \frac{F(s') ds'}{(s' - v)^n (s' - s)} = \int_a^{\infty} \frac{F(s')}{(s' - v - i\varepsilon)^n (s' - s \mp i\varepsilon)} ds' \quad (22)$$

along contours shown in Fig. 8, we find after some elementary operations the following expressions

$$J_2 = \int_a^{\infty} \frac{F(s')}{(s' - v - i\varepsilon)^n (s' - s - i\varepsilon)} ds' = \text{PP} \int_a^{\infty} \frac{F(s')}{(s' - v)^n (s' - s)} ds' - i\pi \left[\frac{F(s)}{(s-v)^n} + \sum_{l=0}^{n-1} F^{(n-l-1)}(v) [(s-v)^{l+1} (n-l-1)!]^{-1} \right], \quad (23)$$

$$\begin{aligned} J_1 &= \int_a^{\infty} \frac{F(s')}{(s' - v - i\varepsilon)^n (s' - s - i\varepsilon)} ds' = \text{PP} \int_a^{\infty} \frac{F(s')}{(s' - v)^n (s' - s)} ds' \\ &\quad + i\pi \left[\frac{F(s)}{(s-v)^n} - \sum_{l=0}^{n-1} F^{(n-l-1)}(v) [(s-v)^{l+1} (n-l-1)!]^{-1} \right]. \end{aligned} \quad (24)$$



FIG. 8

Here the generalized principal-value integral, defined by the limiting procedure of Eq. (19), may also be expressed in terms of principal value integrals

$$\begin{aligned} \text{PP} \int_a^\infty \frac{F(s') ds'}{(s' - \nu)^n (s' - s)} \\ = \sum_{k=0}^{n-2} \frac{F^{(k)}(\nu)}{k!} \left\{ \sum_{l=0}^{n-k-2} \frac{1 - (-1)^{n-k-l-1}}{(s - \nu)^{l+1} (\nu - a)^{n-k-l-1}} \right. \\ + \sum_{l=n-k}^{n-1} \frac{(\nu - a)^{k+l-n+1} [(-1)^{k+l-n+1} - 1]}{(k + l - n + 1)} \\ + \frac{1}{(s - \nu)^{n-k}} \ln \frac{\nu - a - (s - \nu)}{\nu - a + (s - \nu)} \\ + \sum_{l=0}^{k-1} (s - \nu)^{k-1-l-n} (\nu - a)^{l+1} [1 - (-1)^{l+1}] (l + 1)^{-1} \Big\} \\ + P \int_a^{2\nu-a} \left[F(s') - \sum_{k=0}^{n-2} \frac{F^{(k)}(\nu) (s - \nu)^k}{k!} \right] \frac{ds'}{(s' - \nu)^n (s' - s)} \\ + P \int_{2\nu-a}^\infty \frac{F(s') ds'}{(s' - \nu)^n (s' - s)}. \end{aligned} \quad (25)$$

In order to solve Eq. (15) we introduce, following Omnes,^[2] the new function

$$F(z) = \frac{1}{2\pi i} \int_L \frac{h^*(s') ds'}{s' - z} \quad (26)$$

and denote correspondingly

$$\begin{aligned} F(s_+) &= \frac{1}{2\pi i} \int_L \frac{h^*(s') \Phi(s')}{s' - s - i\varepsilon} ds', \\ F(s_-) &= \frac{1}{2\pi i} \int_L \frac{h^*(s') \Phi(s')}{s' - s + i\varepsilon} ds'. \end{aligned} \quad (27)$$

Then the desired function φ can be expressed in terms of $F(z)$ as follows

$$\varphi(s) = [F(s_+) - F(s_-)] / h^*. \quad (28)$$

Equation (15) becomes:

$$e^{-2i\delta} F(s_+) - F(s_-) = h^*(s) f_0(s), \quad f_0(s) = q(s) (s - \nu - i\varepsilon)^{-n}. \quad (29)$$

The solution of Eq. (29) obtained by Omnes^[2] is valid also in the case when the function $f_0(s)$ is singular:

$$\begin{aligned} F(z) &= \Phi(z) \Omega(z), \quad \Omega(z) = e^{u(z)}, \\ u(z) &= \frac{1}{\pi} \int_L \delta_1(z') (z' - z)^{-1} dz', \end{aligned} \quad (30)$$

and if the last integral diverges then we take

$$u(z) = \frac{1}{\pi} \int_L z \delta_1(z') [z' (z' - z)]^{-1} dz'. \quad (31)$$

According to Omnes

$$\Omega(s_+) = e^{\rho + i\delta_1}, \quad \Omega(s_-) = e^{\rho - i\delta_1}, \quad (32)$$

$$\rho(s) = \frac{1}{\pi} P \int \delta_1(s') (s' - s)^{-1} ds', \quad (33)$$

or, if the integral diverges,

$$\rho(s) = \frac{1}{\pi} P \int_a^\infty s \delta_1(s') [s' (s' - s)]^{-1} ds'. \quad (34)$$

Again, as was the case for Omnes, the equation

$$\Phi(s_+) - \Phi(s_-) = f_0(s) \sin \delta_1(s) e^{-\rho(s)} \quad (35)$$

is valid, and the general form of the solution is preserved:

$$\Phi(z) = \frac{1}{2\pi i} \int_L f_0(s') \sin \delta_1(s') e^{-\rho(s')} \frac{ds'}{s' - z}. \quad (36)$$

The difference between our solution and that of Omnes becomes manifest only when we actually use the singular function $f_0(s)$ in the contour integral (36) to evaluate $\Phi(z)$. Since the integrand has a pole of n -th order at the point $s' = \nu$, the integration must be performed according to Eqs. (23), (24) and (25). After substitution of these results into Eq. (28) we obtain the final formula:*

$$\begin{aligned} \varphi(s) &= f_0(s) + \exp[\rho(s)] \\ &+ i\delta_1(s) \left\{ -i \left[\sum_{l=0}^{n-1} [\sin \delta_1(\nu) e^{-\rho(\nu)} q(\nu)]^{(n-l-1)} \right. \right. \\ &\times [(s - \nu)^{l+1} (n - l - 1)!]^{-1} \\ &- q(s) \sin \delta_1(s) e^{-\rho(s)} (s - \nu)^{-n} \Big] + \\ &\left. + \frac{1}{\pi} \text{PP} \int_a^\infty q(s') \sin \delta_1(s') e^{-\rho(s')} \frac{ds'}{(s' - s) (s' - \nu)^n} \right\}. \end{aligned} \quad (38)$$

Here, as above, PP denotes taking the integral in the generalized principal value sense, as defined by Eqs. (19) and (25). Let us point out an important property of the generalized principal value: if the integrand is of the form $F(s') / (s' - \nu)^n (s' - s)$ [with $F(s')$ a regular function], i.e. if the integrand has a pole of n -th order, then the generalized principal value has no poles as a function of s . It follows from here that the solution of the integral equation (15) has a pole of the same order as the inhomogeneous function, and, as can

*After the present work was completed and sent to press the author became familiar with the work of Galanin and Grashin,^[3] in which a method is given for the evaluation of integrals $\Phi(z)$, Eq. (36), in the case when the phase shift δ_l is of the form

$$y' \sqrt{y} \cot \delta_l = \Sigma \alpha_k y^k, \quad y = s / 4\mu^2 - 1. \quad (37)$$

In connection with the appearance of the paper by Galanin and Grashin,^[3] Sec. 4 of the present paper has been revised.

*ctg = cot.

be seen from Eq. (37), in the immediate neighborhood of the pole the coefficients of the pole term in the solution and in the inhomogeneous function are the same. It further follows directly from Eqs. (24) and (25) that already the first iteration of Eq. (15) removes the singularity. This proves the correctness of our choice of the sign of $i\epsilon$ in the inhomogeneous term of Eq. (15).

Equation (15) may be easily generalized to the case when $\varphi(s)$ falls off slowly at infinity, or tends to a constant limit. By introducing into Eq. (14) the function $\Phi(s) = \varphi_1(s)/s$ instead of the function $\varphi_1(s)$ we get

$$\Phi(s) = \frac{f_0(s)}{s} + \frac{1}{\pi} \int_a^\infty e^{-i\delta_1} [s' - s - i\epsilon]^{-1} \sin \delta_1 \Phi ds', \quad (39)$$

i.e. $\varphi_1(s) = s\Phi(s)$, where $\Phi(s)$ — the solution of Eq. (39) — is given by Eq. (38) provided we replace in the latter $q(s)$ by $q(s)/s$.

4. THE AMPLITUDE IN FIRST APPROXIMATION

For simplicity we make the following change of variables

$$x = 1 - 4\mu^2/s, \quad x' = 1 - 4\mu^2/s'. \quad (40)$$

In this notation the solution of Eq. (14) for the electrical part of the amplitude is given in first approximation by

$$\begin{aligned} \varphi_1 = & \frac{-4mGe(1-x)^2}{m^4 \sqrt{x}(x-\alpha)^2} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + \exp[\rho(x) + i\delta_1(x)] \\ & \times \left\{ -\frac{iF'(\alpha)}{(x-\alpha)} + \frac{i[F(x) - F(\alpha)]}{(x-\alpha)^2} \right. \\ & + PP \frac{1}{\pi} \int_0^1 \frac{F(x') dx'}{(x'-x)(x'-\alpha)} \left. \right\} + Ae^{i\delta_1(x)} \left\{ \cos \delta_1(x) \right. \\ & + e^{\rho(x)} \frac{1}{\pi} P \int_0^1 \sin \delta_1(x') e^{-\rho(x')} \frac{dx'}{x'-x} \left. \right\}. \quad (41) \end{aligned}$$

In Eq. (41) we have used the notation:

$$\begin{aligned} F(x) &= \sin \delta_1(x) e^{-\rho(x)} q(x), \quad \alpha = 1 - 4\mu^2/m^2; \\ q(x) &= \frac{-4mGe(1-x)^2}{m^4 \sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}. \quad (42) \end{aligned}$$

The generalized principal value integral appearing in Eq. (41) can be calculated either by the limit procedure, Eq. (19), or by expressing it in a form in which the singularities have already been removed. Using Eq. (20) we obtain

$$\begin{aligned} PP \int_0^1 \frac{F(x') dx'}{(x'-x)(x'-\alpha)^2} &= \frac{1}{(x-\alpha)^2} \left[P \int_{2\alpha-1}^1 \frac{F(x') dx'}{(x'-x)} \right. \\ &- P \int_0^1 \frac{F(x') dx'}{x'-\alpha} \left. \right] + P \int_0^{2\alpha-1} \frac{F(x') dx'}{(x'-x)(x'-\alpha)} - \frac{1}{(x-\alpha)} \\ &\left\{ -\frac{2F(\alpha)}{1-\alpha} + P \int_{2\alpha-1}^1 \frac{[F(x') - F(\alpha)] dx'}{(x'-\alpha)^2} \right\}. \quad (43) \end{aligned}$$

We note that the integrals from 2α to 1 can be neglected.

Below we give expressions for $F(x)$ and $\rho(x)$ for the representation of the P-wave $\pi\pi$ scattering phase shift $\delta_1(x)$ proposed by K. A. Ter-Martirosyan:

$$\delta_1(x) = x^{1/2} [\beta + \gamma x + \dots]. \quad (44)$$

Carrying out calculations according to Eq. (31) we obtain

$$\begin{aligned} \rho(x) = & \frac{2\beta}{\pi} \left(\frac{1}{3} + x + \frac{1}{2} x^{1/2} \ln \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) \\ & + \frac{2\gamma}{\pi} \left(\frac{1}{5} + \frac{x}{3} + x^2 + \frac{1}{2} x^{3/2} \ln \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) + \dots; \quad (45) \end{aligned}$$

$$\begin{aligned} F(x) = & -\sin [x^{1/2}(\beta + \gamma x + \dots)] e^{-\rho(x)} \cdot 4mGe(m^4 \sqrt{x})^{-1} \\ & \times (1-x)^2 \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}. \quad (46) \end{aligned}$$

For the derivative of the function $F(x)$ at the point $x = \alpha$ we get

$$\begin{aligned} F'(\alpha) = & -\frac{2q(\alpha)}{1-\alpha} \sin \delta_1(\alpha) e^{-\rho(\alpha)} \\ & - (1-\alpha)^2 \frac{4Gem}{m^4} e^{-\rho(\alpha)} \left\{ \cos \delta_1(\alpha) \left(\frac{3}{2} \beta + \frac{5}{2} \gamma \alpha + \dots \right) \right. \\ & \times \ln \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} + \sin \delta_1(\alpha) \left[-\frac{1}{2\alpha^{3/2}} \ln \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} + \frac{1}{\alpha(1-\alpha)} \right] \\ & - \sin \delta_1(\alpha) \left[\frac{2\beta}{\pi} \left(\frac{1}{\sqrt{\alpha}} + \frac{3}{4} \ln \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} - \frac{\sqrt{\alpha}}{2(1-\alpha)} \right) \right. \\ & + \frac{2\gamma}{\pi} \left(\frac{1}{3\sqrt{\alpha}} + 2\sqrt{\alpha} + \frac{5}{4} \alpha \ln \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} \right. \\ & \left. \left. - \frac{\alpha^{3/2}}{2(1-\alpha)} \right) \right] \ln \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \left. \right\}. \quad (47) \end{aligned}$$

The results obtained in this work are going to be tabulated on computers. In order, however, to obtain a feeling for the general character of the solution we give below the results of approximating that part of the amplitude, which is connected with radiation from the block of heavy particles.

The integrand in Eq. (41) was expanded in a power series. As a result we obtained (for the case $\gamma = 0, \beta < 1, x < 1$)

$$\begin{aligned} \varphi_A = & Ae^{i\beta x^{1/2}} \left\{ \cos(\beta x^{1/2}) + \left[\frac{2\beta}{\pi} \left(\frac{1}{3} + x + \frac{1}{2} x^{1/2} \ln \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) \right. \right. \\ & + \frac{4\beta^2}{\pi^2} \left(\left(\frac{1}{5} + \frac{x}{3} + x^2 + \frac{1}{2} x^{3/2} \ln \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) \right. \\ & \left. \left. (\sqrt{x} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} - 1) \right) \right. \\ & \left. \left. + \frac{1}{7\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} \right] \left[1 - \frac{2\beta}{\pi} \left(x + \frac{1}{2} x^{1/2} \ln \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) \right]^{-1} \right\}. \quad (48) \end{aligned}$$

For the representation of the phase shift in the form given by Goebel,^[3]

$$\text{tg } \delta_1 = b^{-1}(s/4\mu^2 - 1)^{1/2} = b^{-1}y^{1/2}, \quad y = s/4\mu^2 - 1, \quad (49)^*$$

the calculation of $e^{\rho(s)}$ has been carried out by Ishida et al.^[4] We use instead the method of
 *tg = tan.

Galanin and Grashin^[5] to calculate directly the contour integral Φ_+ in Eq. (36). Making use of a formula from ref^[5], and also of Eq. (19) when evaluating improper integrals, we obtain

$$\varphi_1 = \varphi_A + \varphi_T. \quad (50)$$

Different expressions are obtained for the "vertex" and "bremsstrahlung" parts of the amplitude φ_A and φ_T depending on the value of the quantity b (or $\rho = |b|^{2/3}$).

If $b < 0$, then

$$\frac{\varphi_A}{A} - 1 = \frac{y^3 - \rho y + \rho^2}{b - iy^{3/2}} \left[-\frac{1}{1 + \rho + \rho^2} + \frac{iy^{3/2}}{y^3 - \rho y + \rho^2} \right]. \quad (51)$$

If $b > 0$, then

$$\frac{\varphi_A}{A} - 1 = \frac{1}{b - iy^{3/2}} \left[-\frac{b(1+y)}{1-\rho} + \frac{y+\rho}{1-\rho} + iy^{3/2} \right]. \quad (52)$$

For the part of the amplitude connected with internal bremsstrahlung we obtain:

If $b < 0$

$$\begin{aligned} -(\varphi_T - f_0) \frac{m_4}{4meG} &= \frac{m^4}{16\mu^4} \frac{(1+y)(y^2 - \rho y + \rho^2)}{b - iy^{3/2}} \\ &\left\{ i \left[\frac{q}{(z^3 - \rho z + \rho^2)(z-y)} - \frac{\rho(\rho - 2z)}{(z-y)(z^2 - \rho z + \rho^2)^2} \right. \right. \\ &\quad \left. - \frac{\rho}{(z-y)(z^2 - \rho z + \rho^2)} \right. \\ &\quad \left. + \frac{y}{(y-z)\sqrt{1+z}(y^2 - \rho y + \rho^2)} \ln \frac{\sqrt{1+y} + \sqrt{y}}{\sqrt{1+y} - \sqrt{y}} \right] \\ &\quad - \operatorname{Im} \left[\frac{2u}{\sqrt{1+u}(u-y)(u-u^*)(u-z)^2} \ln \frac{\sqrt{1+u} + \sqrt{u}}{\sqrt{1+u} - \sqrt{u}} \right. \\ &\quad \left. - \frac{2u^*\pi}{\sqrt{1+u^*}(u^*-y)(u^*-u)(u^*-z)^2} \right] \\ &\quad - \pi \left[\frac{z}{\sqrt{1+z}(z^2 - \rho z + \rho^2)(y-z)^2} \right. \\ &\quad \left. + \frac{\rho^2 - z^2 + z(-3z^2 + \rho z + \rho^2)/2}{(y-z)(1+z)\sqrt{1+z}(z^2 - \rho z + \rho^2)^2} \right. \\ &\quad \left. - \frac{y}{(y-z)^2\sqrt{1+y}(y^3 - \rho y + \rho^2)} \right] \Big\}, \quad (53) \end{aligned}$$

where we have used the following abbreviations

$$z = m^2/4\mu^2 - 1, \quad u = \rho e^{+i\pi/3}, \quad u^* = \rho e^{-i\pi/3},$$

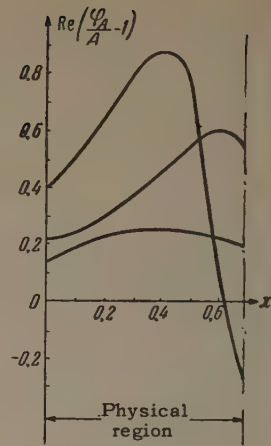
$$\rho = \frac{z}{\sqrt{1+z}} \ln \frac{\sqrt{1+z} + \sqrt{z}}{\sqrt{1+z} - \sqrt{z}},$$

$$q = \frac{2+z}{2(1+z)\sqrt{1+z}} \ln \frac{\sqrt{1+z} + \sqrt{z}}{\sqrt{1+z} - \sqrt{z}} + \frac{\sqrt{z}}{1+z}.$$

If $b > 1$, then

$$\begin{aligned} -(\varphi_T - f_0) \frac{m^4}{4meG} &= \frac{m^4}{16\mu^4} \frac{(1+y)(y+\rho)}{b - iy^{3/2}} \left\{ i \left[\frac{q}{(z-y)(z+\rho)} \right. \right. \\ &\quad \left. - \frac{\rho}{(z-y)(z+\rho)^2} - \frac{\rho}{(z+\rho)(z-y)^2} \right] \\ &\quad + \frac{y}{(y+\rho)(y-z)^2\sqrt{1+y}} \ln \frac{\sqrt{1+y} + \sqrt{y}}{\sqrt{1+y} - \sqrt{y}} \Big\} \\ &\quad + \frac{1}{\sqrt{\rho-1}(y+\rho)(\rho+z)^2} \ln \frac{\sqrt{\rho-1} + \sqrt{\rho}}{\sqrt{\rho-1} - \sqrt{\rho}} \end{aligned}$$

FIG. 9. Comparison of the solutions for the vertex part of the amplitude for two representations of the phase shift δ_1 : lower curve — $\beta = 0.614$ (expansion in the parameter $x = 1 - 4\mu^2/s$); upper and middle curves — b equal respectively to 1.5 and 5 [expansion in the parameter $y = s/4\mu^2 - 1 = x/(1-x)$].



$$\begin{aligned} &+ \frac{\pi y}{(y-z)^2(y+\rho)\sqrt{1+y}} - \frac{2\pi}{\sqrt{1+z}(z+\rho)(y-z)^2} \\ &+ \frac{\pi}{2} \frac{(z\rho - z^2 + 2\rho)}{(z+\rho)^2(1+z)^{3/2}z-y} \Big\}. \quad (54) \end{aligned}$$

The case $1 > b > 0$, which corresponds to very large phase shifts at the borders of the physical region (in excess of 70°), has not been considered. Some of the cases are shown graphically in Fig. 9.

5. AN ESTIMATE OF THE CONTRIBUTION OF THE NEGLECTED TERMS

To prove the validity of the first approximation it is necessary to make an estimate of the second approximation, in which the $K\pi$ interaction is taken into account. After averaging over s we obtain the following equation for the second order correction $\varphi_2 = \varphi - \varphi_1$, where φ is the exact solution of Eq. (8) subject to condition (14),

$$\begin{aligned} \varphi_2 &= \frac{1}{\pi} \int_{4\mu^2}^{\infty} \exp\{i\delta_1(s')\} \sin \delta_1(s') \varphi_2(s') \left[\frac{1}{s' - s - i\epsilon} \right. \\ &\quad \left. - \frac{1}{s'} \right] ds' + \Psi(s); \quad (55) \end{aligned}$$

$$\begin{aligned} \Psi(s) &\sim \frac{2}{\pi} \int_{(m+\mu)^2}^{\infty} \bar{\varphi}_0(t') \exp\{-i\delta_1'(t')\} \sin \delta_1'(t') \left[L(t', s) \right. \\ &\quad \left. - \frac{1}{t'} \right] dt'. \quad (56) \end{aligned}$$

Here the function $\bar{\varphi}_0(t')$ is obtained by averaging the pole term in the amplitude $4meG(s - m^2)^{-1} \times (t - \mu^2)^{-1}$ over s at a given t .

The ratio of φ_2 to the pole term is no larger than the ratio of the function $\Psi(s)$ to the pole term. After taking into account that $|\exp\{-i\delta_1'\} \sin \delta_1'| < 1$, we get from formula (56) the following condition for the second order connection to be small compared to the pole term:

$$\frac{E_\gamma^2(E_{\gamma \max} - E_\gamma)^{3/2}}{m^3 \sqrt{m - 2E_\gamma}} \ln^{-1} \left(\frac{\sqrt{m/2 - E_\gamma} + \sqrt{E_{\gamma \max} - E_\gamma}}{\sqrt{m/2 - E_\gamma} - \sqrt{E_{\gamma \max} - E_\gamma}} \right) \ll 1, \quad (57)$$

where E_γ is the photon energy. The condition (57) is certainly fulfilled for $E_\gamma = 0$; $E_\gamma = E_{\gamma\max}$ or $s = 4\mu^2$; $s = m^2$.

Consequently the most unfavorable case corresponds to the midpoint $E_\gamma = E_{\gamma\max}/2 = (m^2 - 4\mu^2)/2m$. At that point the expression on the left is equal to $1/80$, i.e., the corrected amplitude is valid to an accuracy of $\sim 1\%$. One obtains a correction of the same order if one considers the channel with four pions. The reason why the approximation is so good has to do with the fact that we are seeking in the first place a 10% correction to the pole term. However such high accuracy presupposes a precise knowledge of the P-wave $\pi\pi$ -interaction phase shift in a large region, including the singular point $s = m^2$ too. For s of the order of $20\mu^2$ the contribution of the four pion channel, which produces a branch point in the amplitude at $s = 16\mu^2$, is unimportant.

Thus, although the four pion channel does not upset things, and a solution of high accuracy can be obtained provided the P-wave phase shift is known with precision, large errors can nevertheless be introduced in the process of substituting concrete expansions of the P-wave phase shift. For this reason of the two solutions proposed above the first one should be preferred, because when the expansion proposed by Ter-Martirosyan is used many terms can be included without par-

ticular difficulties. The first of the neglected terms contains $x^{7/2}$. At $s \sim m^2$ we have $x^{7/2} \sim 1/5$. And if it should later turn out that a larger number of terms is needed to describe the P wave phase shift then our solution can be extended to cover that case without difficulty.

We gave above estimates of the contribution of the $K\pi$ channels. A 1% correction was found without taking into account the smallness of the δ'_1 phase shift. In actuality the contribution from other channels with small phase shifts is smaller by another order of magnitude, i.e., is of the order of 0.1% in comparison with the pole term.

The author expresses his deep gratitude to K. A. Ter-Martirosyan for suggesting this problem and for discussions, and to A. F. Grashin and L. B. Okun' for discussions.

¹J. D. Good, Phys. Rev. **113**, 352 (1959).

²R. Omnes, Nuovo cimento **8**, 316 (1958).

³C. Goebel, Phys. Rev. Lett. **1**, 337 (1958).

⁴Ishida, Takahashi, and Ueda, Progr. Theoret. Phys. **23**, 731 (1960).

⁵A. D. Galanin and A. F. Grashin, JETP **41**, 633 (1961), Soviet Phys. JETP **14**, 454 (1961).

Translated by A. M. Bincer

MINIMUM NUMBER OF PARTIAL WAVES IN REACTIONS IN WHICH THERE ARE SEVERAL PARTICLES IN THE FINAL STATE

HSIEN TING-CH'ANG and CH'EN TS'UNG-MO

Joint Institute for Nuclear Research

Submitted to JETP editor March 4, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 784-789 (September, 1961)

A universal inequality is derived for the minimum number of partial waves involved in a reaction in which more than two particles occur in the final state.

1. INTRODUCTION

THE minimum number $L_{\min}$ of partial waves occurring in collisions of particles at high energies has been determined by several authors. Rarita and Schwed^[1] have shown how to calculate $L_{\min}$ for elastic scattering, given a knowledge of the total cross section of the interaction. Recently Grishin and Ogievetskii^[2] have derived an inequality which is very effective in the determination of the minimum number of partial waves in two-particle reactions, if one knows the total elastic cross section and the differential cross section for certain angles.

As is well known, most high-energy collision processes are multiple processes in which more than two particles appear in the final state. The question arises as to how to determine the minimum number of partial waves in such collisions. A knowledge of $L_{\min}$ is important for the processing of experimental results, since $L_{\min}$ is connected with the minimum interaction radius.

In the present paper the inequality obtained by Grishin and Ogievetskii for two-particle reactions is extended to the case of reactions in which more than two particles appear in the final state. It involves the angular distribution of one of the particles in the final state, the total cross section for the given channel, and $L_{\min}$. In Sec. 2 we discuss in detail the choice of the independent variables for the description of the final states of a three-particle system. In Sec. 3 the inequality is obtained for the case of particles without spin ($0 + 0 \rightarrow 0 + 0 + 0$). In Sec. 4 it is shown that the inequality obtained in Sec. 3 can be carried over without change to the cases

$$0 + \frac{1}{2} \rightarrow 0 + 0 + \frac{1}{2}, \quad \frac{1}{2} + \frac{1}{2} \rightarrow 0 + \frac{1}{2} + \frac{1}{2},$$

$$\frac{1}{2} + \frac{1}{2} \rightarrow 0 + 0 + 0, \quad 0 + 0 \rightarrow 0 + \frac{1}{2} + \frac{1}{2}.$$

In Sec. 5 the inequality for the case of three par-

ticles in the final state is extended to the case in which there are n particles in the final state; in Sec. 6 applications of this inequality are discussed.

2. KINEMATICS OF THE THREE-PARTICLE SYSTEM

To describe the states of a three-particle system we shall introduce three other vectors instead of the momentum vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ of the particles. As the first we take the momentum of the center of mass, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$; in the center-of-mass system of the three particles (which we shall hereafter call simply the 3c system), $\mathbf{P} = 0$. As the second we choose $\mathbf{P}_{3c}$ ($|\mathbf{P}_{3c}|, \Omega_{3c}$), the momentum in the 3c system of that one of the three particles which can be identified experimentally (for example, the recoil nucleon, or K meson, or hyperon). In the system in which the other two particles, taken as a whole, are at rest (we shall call it the 2c system), these two particles move in opposite directions, so that as the third vector we can take the direction Ω_{2c} of the relative momentum of the two particles in the 2c system and the energy $\mathfrak{M}_{2c}$ of these particles in the 2c system.

This choice of the independent variables has a number of advantages. First, as is shown in the Appendix, the integration over the phase space can be separated into two parts:

$$\begin{aligned} & \iiint \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{8 E_1 E_2 E_3} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{P}_i) \delta(E_1 + E_2 + E_3 - E_i) \\ &= \int d\mathfrak{M}_{2c}^2 G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) \iint d\Omega_{2c} d\Omega_{3c}. \end{aligned} \quad (1)$$

Here $\mathfrak{M}_{3c}$ is the total energy of the three particles in the 3c system, and $\mathbf{P}_i$ and E_i are the momentum and energy in the initial state. The function $G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2)$ does not depend on the angles, and the limits of the integrations over $d\Omega_{2c}$ and $d\Omega_{3c}$ do not depend on each other. Thus the integrations over the angles and the energies are separated.

Second, starting from the work of Chou Kuang-chao and Shirokov,^[3] one can show that the total angular momentum of the three particles can be obtained in the usual way by compounding l_{2c} , the relative angular momentum of the two particles in the 2c system, and l_{3c} , the angular momentum of the identified particle relative to the two other particles in the 3c system.

3. INTERACTIONS OF THE TYPE

$$0 + 0 \rightarrow 0 + 0 + 0$$

Let us first consider the simplest case, in which all of the particles are spinless ($0 + 0 \rightarrow 0 + 0 + 0$). Let us choose the variables as indicated in Section 2; then the general form of the amplitude for the process is

$$F = \sum R_L^{l_{2c}, l_{3c}} Y_{L, M}^*(\Omega_i) C_{L, M}^{l_{2c}, l_{3c}, 0} Y_{l_{2c}, M}(\Omega_{2c}) \times Y_{l_{3c}, 0}(\Omega_{3c}). \quad (2)$$

Here the summation is taken over all quantum numbers. The function $R_L^{l_{2c}, l_{3c}}$ depends on the total angular momentum L of the three-particle system, and on l_{2c} and l_{3c} , the angular momenta mentioned at the end of the preceding section. The other arguments of the function $R_L^{l_{2c}, l_{3c}}$ are invariant under Lorentz transformations. $Y_{l, m}$ are spherical harmonics; Ω_i is the unit vector in the direction of the momentum of the incident particle in the 3c system; $C_{L, M}^{l_{2c}, l_{3c}, 0}$ are Clebsch-Gordon coefficients.

If we choose the z axis in the direction Ω_{3c} , Eq. (2) can be written in the form

$$F = \sum R_L^{l_{2c}, l_{3c}} \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} Y_{l_{2c}, M}(\Omega_{2c}) Y_{L, M}^*(\Omega_i). \quad (3)$$

The angular distribution of the identified particle is of the form

$$\sigma(\theta) = \int G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2 \int |F|^2 d\Omega_{2c} = \sum_{M, l_{2c}} \int \left| \sum_{L, l_{3c}} R_L^{l_{2c}, l_{3c}} \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} Y_{L, M}^*(\theta) \right|^2 \times G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2 \quad (4)$$

and the cross section for the process is given by the formula

$$\sigma_3 = \int \sigma(\theta) d\Omega = \sum_{L, M, l_{2c}} \int G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2 \left| \sum_{l_{3c}} \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} \times R_L^{l_{2c}, l_{3c}} C_{L, M}^{l_{2c}, M; l_{3c}, 0} \right|^2. \quad (5)$$

Let us assume that in the expressions (4) and (5) we can confine ourselves to a finite number L_{min} of partial waves. Then by using the Cauchy inequality

$$\sum |A_i B_i|^2 \leq \sum |A_i|^2 \sum |B_i|^2,$$

we get

$$\sigma(\theta) \leq \sum_{M, l_{2c}} \int \left| \sum_{L, l_{3c}} R_L^{l_{2c}, l_{3c}} \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} \right|^2 \times \sum_L |Y_{L, M}(\theta)|^2 G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2 \leq \sum_{L, M, l_{2c}} \int \left| \sum_{l_{3c}} R_L^{l_{2c}, l_{3c}} \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} \right|^2 \times G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2 \sum_{LM} |Y_{L, M}(\theta)|^2 = \sigma_3 \sum_{L=0}^{L_{min}} \frac{2L+1}{4\pi} \quad (6)$$

or

$$4\pi\sigma(\theta) / \sigma_3 \leq (L_{min} + 1)^2. \quad (7)$$

It must be noted that after the summation over M the right member of Eq. (6) already does not depend on θ ; this is unlike the case of the other two particles, for which there is no such summation over M and the right side of the inequality depends on θ .^[2]

4. INTERACTIONS OF THE TYPE

$$0 + \frac{1}{2} \rightarrow 0 + 0 + \frac{1}{2}$$

Let us consider the case $0 + \frac{1}{2} \rightarrow 0 + 0 + \frac{1}{2}$. The amplitude for this process is given by the expression

$$F = \sum R_{JLL'}^{l_{2c}, l_{3c}} C_{J, M}^{L', M-\beta; \frac{1}{2}, \beta} C_{L', M-\beta}^{l_{3c}, 0; l_{2c}, M-\beta} C_{J, M}^{L, M-\alpha; \frac{1}{2}, \alpha} \times \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} Y_{l_{2c}, M-\beta}(\Omega_{2c}) Y_{L, M}^*(\Omega_i); \quad (8)$$

here the summation is over J, L, L', M, l_{2c} , and l_{3c} , where J is the total angular momentum of the three-particle system in the 3c system, L and L' are the total orbital angular momenta, and α and β are the spin projections of the particles in the initial and final states in the 3c system. The other symbols are the same as in the preceding section.

The angular distribution of the identified particle takes the form

$$\sigma(\theta) = \sum_{\alpha\beta} \int G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2 \int |F|^2 d\Omega_{2c} = \sum_{\alpha, \beta} \int \left| \sum_{L, L', J} R_{JLL'}^{l_{2c}, l_{3c}} C_{J, M}^{L', M-\beta; \frac{1}{2}, \beta} C_{L', M-\beta}^{l_{3c}, 0; l_{2c}, M-\beta} \times C_{J, M}^{L, M-\alpha; \frac{1}{2}, \alpha} \left(\frac{2l_{3c} + 1}{4\pi} \right)^{1/2} Y_{L, M}^*(\theta) \right|^2 G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) d\mathfrak{M}_{2c}^2. \quad (9)$$

Using the Cauchy inequality and making calculations like those given in Sec. 3, we get the inequality

$$4\pi\sigma(\theta) / \sigma_3 \leq (L_{min} + 1)^2, \quad (10)$$

where

$$\sigma_3 = \int \sigma(\theta) d\Omega. \quad (11)$$

The inequality (10) is of the same form as for the spinless case. It is easy to show that this same result can also be obtained for the cases

$$\begin{aligned} 1/2 + 1/2 \rightarrow 0 + 1/2 + 1/2, \quad 1/2 + 1/2 \rightarrow 0 + 0 + 0, \\ 0 + 0 \rightarrow 0 + 1/2 + 1/2 \end{aligned}$$

5. INTERACTIONS IN WHICH THERE ARE n PARTICLES IN THE FINAL STATE

Let us first consider the case $0 + 0$

$\rightarrow \overbrace{0 + \dots + 0}^n$. In choosing the variables for this case we can take first the momentum $\mathbf{P}$ of the center of mass of the n particles. In the center-of-mass system of these n particles (the nc system), $\mathbf{P} = 0$. Next, $\mathbf{P}_{nc}$ ($|\mathbf{P}_{nc}|, \Omega_{nc}$) is the momentum of the identified particle in the nc system. In the system in which the $n-1$ particles are at rest taken as a whole [the $(n-1)c$ system], it is convenient to take as the variables $\mathfrak{M}_{(n-1)c}$, the total energy of the $n-1$ particles in the $(n-1)c$ system, and $\Omega_{(n-1)c}$, the unit vector in the direction of the relative momentum $\mathbf{P}_{(n-1)c}$ of one of these $n-1$ particles and the other $n-2$ particles, taken as a whole, in the $(n-1)c$ system. The remaining variables are chosen analogously, and are written $\mathfrak{M}_{(n-2)c}^2, \Omega_{(n-2)c}, \dots, \mathfrak{M}_{2c}^2, \Omega_{2c}$. The advantages of this choice of the independent variables have been indicated in Section 2.

The integration over the phase space is of the form

$$\begin{aligned} \int \frac{dp_1 dp_2 \dots dp_n}{2^n E_1 E_2 \dots E_n} \delta(p_1 + p_2 + \dots + p_n - p_i) \delta(E_1 + \dots \\ + E_n - E_i) = \int G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2, \dots, \mathfrak{M}_{nc}^2) d\mathfrak{M}_{2c}^2 \dots d\mathfrak{M}_{(n-1)c}^2 \\ \times \int d\Omega_{2c} \dots d\Omega_{nc}, \end{aligned} \quad (12)$$

where G is a function which depends only on the energies and not on the angles.

By means of the set of independent variables that has been indicated one can reduce the amplitude for the process to the form

$$\begin{aligned} F_n = \sum R_{L, L_{3c}, \dots, L_{(n-1)c}}^{l_{2c}, l_{3c}, \dots, l_{nc}} C_{L_{3c}, M_{3c}}^{l_{2c}, l_{3c}, l_{3c}, l_{3c}} C_{L_{4c}, M_{4c}}^{L_{3c}, M_{3c}, l_{4c}, l_{4c}} \dots \\ \times C_{L, M}^{L_{(n-1)c}, M_{(n-1)c}, l_{nc}, l_{nc}} Y_{L, M}^*(\Omega_i) Y_{l_{2c}, l_{2c}}(\Omega_{2c}) \dots \\ \times Y_{l_{nc}, l_{nc}}(\Omega_{nc}), \end{aligned} \quad (13)$$

where L and M are the total angular momentum of all n particles in the nc system and the z component of this angular momentum; l_{ic} and μ_{ic} are the relative angular momentum and z component of angular momentum that correspond to the momentum $\mathbf{p}_{ic}$ in the ic system; L_{ic} and M_{ic}

are the total angular momentum and its z component in the ic system; the summation is taken over all the angular momenta and z components.

Let us take the z axis along Ω_{nc} ; then the angular distribution of the identified particle and the total cross section can be written in the forms

$$\begin{aligned} \sigma_n(\theta) = \int |F_n|^2 G(\mathfrak{M}_{2c}^2, \dots, \mathfrak{M}_{nc}^2) d\mathfrak{M}_{2c}^2 \dots d\mathfrak{M}_{(n-1)c}^2 d\Omega_{2c} \dots \\ \times d\Omega_{(n-1)c}, \end{aligned} \quad (14)$$

$$\sigma_n = \int \sigma_n(\theta) d\Omega. \quad (15)$$

Using the Cauchy inequality, we get from Eqs. (14) and (15)

$$4\pi\sigma_n(\theta) / \sigma_n \leq (L_{min} + 1)^2. \quad (16)$$

This inequality is analogous to the one obtained in the case of three particles without spin. For the general cases in which several particles have spins different from zero it can be shown in just the same way that the result is the same as in the case of spinless particles.

6. DISCUSSION

The inequality (16) is useful for the determination of L_{min} . It enables us to calculate L_{min} if we know the total cross section for a process in which a definite number of particles emerge in the final state and the angular distribution of one identified particle (K meson, hyperon, recoil nucleon, or antibaryon) in the c.m.s.

In view of the fact that the right member of Eq. (16) does not depend on θ , the inequality is to be written in the form

$$4\pi(\sigma_n(\theta))_{max} / \sigma_n \leq (L_{min} + 1)^2. \quad (17)$$

It must be noted that if we take the z axis in the direction Ω_i , then between the angular distribution of one identified particle out of the j particles in the jc system

$$\begin{aligned} \sigma_n(\theta_j) = \int |F_n|^2 G(\mathfrak{M}_{2c}^2, \dots, \mathfrak{M}_{nc}^2) d\mathfrak{M}_{2c}^2 \dots d\mathfrak{M}_{(n-1)c}^2 \\ \times d\Omega_{2c} \dots d\Omega_{(j-1)c} d\Omega_{(j+1)c} \dots d\Omega_{nc} \end{aligned} \quad (18)$$

and the cross section for the process we have the relation

$$4\pi\sigma_n(\theta_j) / \sigma_n \leq [(l_j)_{min} + 1]^2. \quad (19)$$

The inequality (19) makes it possible to determine the minimum number of partial waves essential for the description of the subsystem jc .

The writers thank Chou Kuang-chao and V. I. Ogievetskii for helpful discussions.

APPENDIX* where

For the case of an n -particle process the integral over phase space is of the form

$$I = \int \frac{d\mathbf{p}_1 \dots d\mathbf{p}_n}{2^n E_1 E_2 \dots E_n} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \dots + \mathbf{p}_n - \mathbf{p}_i) \delta(E_1 + \dots + E_n - E_i). \quad (\text{A.1})$$

It can be written in the explicitly invariant form

$$I = \int d^4 p_1 \dots d^4 p_n \delta(p_1^2 + m_1^2) \dots \delta(p_1^2 + m_n^2) \delta^4(p_1 + p_2 + \dots + p_n - p_i). \quad (\text{A.2})$$

After the transformation

$$p_1 + p_2 = k_2, \quad p_1 - p_2 = 2q_2$$

we get

$$I = \int d^4 k_2 d^4 q_2 d^4 p_3 \dots d^4 p_n \delta\left(\left(\frac{k_2}{2} + q_2\right)^2 + m_1^2\right) \delta\left(\left(\frac{k_2}{2} - q_2\right)^2 + m_2^2\right) \delta(p_3^2 + m_3^2) \dots \delta(p_n^2 + m_n^2) \times \delta^4(k_2 + p_3 + \dots + p_n - p_i). \quad (\text{A.3})$$

Noting that

$$d^4 q_2 = \frac{1}{2} \sqrt{q_2^2 + q_{20}^2} dq_{20} dq_2^2 d\Omega_2,$$

we can perform the integration over $d^4 q_2$ in the $2c$ system, in which

$$k_{2c}^2 = -k_{2c0}^2, \quad (k_{2c} q_{2c}) = -k_{2c0} q_{2c0}$$

and Eq. (A.3) then takes the form

$$I = \left(\frac{1}{2}\right)^3 \int d\mathfrak{M}_{2c}^2 d\Omega_{2c} d^4 k_{2c} d^4 p_3 \dots d^4 p_n \left(1 - 2 \frac{m_1^2 + m_2^2}{\mathfrak{M}_{2c}^2} + \left(\frac{m_1^2 - m_2^2}{\mathfrak{M}_{2c}^2}\right)^{1/2}\right) \delta(k_{2c}^2 + \mathfrak{M}_{2c}^2) \delta(p_3^2 + m_3^2) \dots \delta(p_n^2 + m_n^2) \times \delta^4(k_{2c} + p_3 + \dots + p_n - p_i),$$

*An analogous method of integration over phase space has been proposed earlier by Kopylov^[4] for the calculation of statistical weights and distributions in theories of multiple production.

$$\mathfrak{M}_{ic}^2/4 = q_{2c}^2 + (m_1^2 + m_2^2)/2.$$

It is easy to see that $\mathfrak{M}_{2c}$ is the total energy of the two particles in the $2c$ system.

Continuing the indicated procedure, i.e., setting

$$k_i + p_{i+1} = k_{i+1}, \quad k_i - p_{i+1} = 2q_{i+1}, \quad i = 3, 4, \dots, n-1$$

and integrating over $d^4 q_{i+1}$ in the $(i+1)c$ system, we finally obtain the integral over phase space in the form

$$I = \int G(\mathfrak{M}_{2c}^2, \dots, \mathfrak{M}_{nc}^2) d\Omega_{2c} \dots d\Omega_{nc} d\mathfrak{M}_{2c}^2 \dots d\mathfrak{M}_{(n-1)c}^2 \quad (\text{A.4})$$

($\mathfrak{M}_{ic}$ is the total energy of particle i in the ic system), where

$$\frac{1}{4} \mathfrak{M}_{ic}^2 = q_{ic}^2 + \frac{1}{2} (\mathfrak{M}_{(i-1)c}^2 + m_i^2), \quad \mathfrak{M}_{1c} = m_1, \\ G(\mathfrak{M}_{2c}^2, \dots, \mathfrak{M}_{nc}^2) = \left(\frac{1}{2}\right)^{(n-1)} \prod_{i=2}^n \left[1 - 2 \frac{\mathfrak{M}_{(i-1)c}^2 + m_i^2}{\mathfrak{M}_{ic}^2} + \left(\frac{\mathfrak{M}_{(i-1)c}^2 - m_i^2}{\mathfrak{M}_{ic}^2}\right)^{2/2}\right].$$

¹ W. Rarita and P. Schwed, Phys. Rev. 112, 271 (1958).

² V. G. Grishin and V. I. Ogievetskii, Nuclear Phys. 18, 516 (1960).

³ Chou Kuang-Chao and M. I. Shirokov, JETP 34, 1230 (1958), Soviet Phys. JETP 7, 851 (1958).

⁴ G. I. Kopylov, JETP 39, 1091 (1960), Soviet Phys. JETP 12, 761 (1961).

COORDINATE FRACTIONAL PARENTAGE COEFFICIENTS FOR MULTISHELL CONFIGURATIONS

I. G. KAPLAN

Institute of Chemical Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor March 10, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 790-799 (September, 1961)

Expressions are found for the coordinate fractional parentage coefficients of an arbitrary multishell configuration in the form of a product of single-shell parentage coefficients and matrix elements of the transformation matrices of the permutation group and the rotation group. Formulas are presented for the matrix elements of the operators F and G in the case when they do not contain the spin variables.

THE problem of computing the energy and other characteristics of a system of n identical particles meets with difficulties in constructing the wave function and in the complexity of the computations. The introduction of fractional parentage coefficients in the papers of Racah^[1] definitely simplified and systematized such computations. Later a whole series of papers appeared on the calculation of fractional parentage coefficients for identical particles (for a single shell).^[2-7] Parentage coefficients for a configuration of several shells were obtained by Levinson,^[8] and an expression for the fractional parentage coefficients of a two-shell configuration was used earlier by Elliott and Flowers.^[9] Balashov, Tumanov, and Shirokov^[10] obtained formulas for the matrix elements of configurations consisting of several shells by applying the methods of second quantization.

If the operators do not contain the spin variables, then to compute their matrix elements we need only know the parentage coefficients of the coordinate wave function. In the present paper we derive formulas for the coordinate fractional parentage coefficients of an arbitrary multishell configuration. We show that these coefficients are expressible as products of coordinate fractional parentage coefficients for a single shell and matrix elements of the transformation matrices of the permutation group and the rotation group. In addition we give formulas for the matrix elements of single-particle and two-particle operators which depend only on the spatial coordinates.

1. TWO-SHELL CONFIGURATION

The primary interest in quantum mechanical calculations is the computation of matrix elements of operators of the type^[11]

$$F = \sum_{i=1}^n f_i, \quad G = \sum_{i < j}^n g_{ij}.$$

It is therefore convenient to represent the wave function of a system of n particles as an expansion in which one or two particles are separated out; the coefficients in such an expansion are called fractional parentage coefficients. We write the formal expansion of the coordinate wave function for a configuration consisting of two shells.*

a) Removal of a single particle, parentage coefficients of the type $\langle n-1, 1 | \rangle n$:

$$\begin{aligned} \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2) [\lambda] (r) LM) \\ = \sum_{\lambda'_2} \sum_{\alpha'_1 L'_1} \varphi((l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2) [\lambda'] (r') L', l_1; LM) \\ \times \langle (l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2) [\lambda'] L', l_1; L | \rangle l_1^{n_1} [\lambda_1] \\ \times \alpha_1 L_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2; [\lambda] L \rangle \\ + \sum_{\lambda'_2} \sum_{\alpha'_2 L'_2} \varphi((l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2) [\lambda'] (r') L', l_2; LM) \\ \times \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2) [\lambda'] L', l_2; L | \rangle l_1^{n_1} [\lambda_1] \\ \times \alpha_1 L_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2; [\lambda] L \rangle. \end{aligned} \quad (1)$$

The function φ is formed as follows: we vector couple the wave function of n 'th particle to the wave function with permutation symmetry $[\lambda'] (r')$ for the configuration of $n-1$ particles. There is no summation over $[\lambda'] (r')$ since the assignment of the Yamanouchi symbol (r)

*Throughout the paper we use the notation of our previous work,^[12] references to which are indicated by a Roman numeral I. The quantum numbers of the configuration left after removal of a particle are indicated by a prime, those of the removed particle by a double prime.

uniquely determines the symmetry of the $(n-1)$ particles (cf. [2-3]).

b) Removal of two particles, parentage coefficients of the type $\langle n-2, 2 | n \rangle$:

$$\begin{aligned} \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2) [\lambda] (r' r'') LM) \\ = \sum_{\lambda'_1} \sum_{\alpha'_1 L'_1 L''_1 L'} \Phi((l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2) [\lambda'] \\ \times (r') L', l_1^2 [\lambda''] L''_1; LM) \\ \times \langle (l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2) [\lambda'] L', \\ \rightarrow l_1^2 [\lambda''] L''_1; L | \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2; [\lambda] L \rangle \\ + \sum_{\lambda'_2} \sum_{\alpha'_2 L'_2 L''_2 L'} \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2-2} [\lambda'_2] \alpha'_2 L'_2) [\lambda'] (r') L', \\ \times l_2^2 [\lambda''] L''_2; LM) \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2-2} [\lambda'_2] \alpha'_2 L'_2) [\lambda'] L', \\ \rightarrow l_2^2 [\lambda''] L''_2; L | \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2} [\lambda_2] \alpha_2 L_2; [\lambda] L \rangle \\ + \sum_{\lambda'_1 \lambda'_2} \sum_{\alpha'_1 L'_1 \alpha'_2 L'_2 L''_1 L''_2} \Phi((l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2) [\lambda'] \\ \times (r') L', l_1 l_2 [\lambda''] L''; LM) \langle (l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2) \\ \times [\lambda'] L', \rightarrow l_1 l_2 [\lambda''] L''; L | \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2} \\ \times [\lambda_2] \alpha_2 L_2; [\lambda] L \rangle. \end{aligned} \quad (2)$$

The choice of the representation $[\lambda']$, when reduced to the subgroup $S_{n-2} \times S_2$, corresponds to assigning the symmetry of $[\lambda'] (r')$ and $[\lambda'']$, so there is no summation over these symbols.

To find the fractional parentage coefficients defined in (1) and (2) we shall start from formula I (20). Our problem is to transform it to the form (1) and (2).

Let us go through the derivation for the case of removal of two particles. For $(r)^A$ in I (20) we choose $(r' r'')$. Just as in [8], we divide the sum over Q into four sums, according to the effect of the permutations on the coordinates of the $(n-1)$ -st and n -th particles: we denote by Q_1 those permutations which bring $n-1$ and n to the positions n_1-1 and n_1 , by Q_2 those which bring $n-1$ to the position n_1 , by Q_3 those which bring n to n_1 , and by Q_4 those permutations which leave $n-1$ and n in their places. It is obvious that

$$n(Q_1) = \frac{(n-2)!}{(n_1-2)! n_2!}, \quad n(Q_2) = n(Q_3) = \frac{(n-2)!}{(n_1-1)! (n_2-1)!},$$

$$n(Q_4) = \frac{(n-2)!}{n_1! (n_2-2)!}, \quad n(Q) = \sum_{i=1}^4 n(Q_i) = \frac{n!}{n_1! n_2!}. \quad (3)$$

Every permutation Q_i can be represented in the form of a product $Q'_i P_i$, where P_i brings $n-1$

and n to the appropriate places, while Q'_i does not act on them, i.e. $Q'_i \in S_{n-2}$. It is not difficult to show that we can choose the following permutations for P_i *

$$P_1 = \begin{cases} P_{n, n-2 \dots n_1-1, n-1, n-3 \dots n_1} & n_2 \text{ odd,} \\ P_{n, n-2 \dots n_1} P_{n-1, n-3 \dots n_1-1} & n_2 \text{ even;} \end{cases} \quad (4)$$

$$P_2 = P_{n-1, n-2 \dots n_1}, \quad P_3 = P_{n, n-1} P_2, \quad P_4 = 1.$$

Let us consider the sum over Q_1 . When transformed appropriately, this sum should give explicit expressions for the fractional parentage coefficients of the type $\langle l_1^{n_1-2} l_2^{n_2}, l_1^2 | l_1^{n_1} l_2^{n_2} \rangle$. After applying P_1 to φ , we have

$$\begin{aligned} \sum_1 = c \sum_{r_1 r_2} \sum_{Q'_1} \langle [\lambda] (r' r'') | Q_1 | [\lambda] (r_1 r_2) \rangle \\ \times Q'_1 \varphi'(l_1^{n_1} [\lambda_1] (r_1) \alpha_1 L_1, l_2^{n_2} [\lambda_2] (r_2) \alpha_2 L_2; \\ \rightarrow LM | 1, 2, \dots, n_1-2, n-1, n; n_1-1, n_1, \dots, n-2); \\ c = \left[\frac{f_\lambda}{f_{\lambda_1} f_{\lambda_2}} \frac{n_1! n_2!}{n!} \right]^{1/2}. \end{aligned} \quad (5)$$

The arrangement of the particle numbers in the argument of the function φ' corresponds to the distribution of the particles over the shells. Since we sum over the Yamanouchi symbols, the expression (5) can be replaced by the equivalent expression

$$\begin{aligned} \sum_1 = c \sum_{\lambda'_1 \lambda'_2 \lambda'_1 r_1 Q'_1} \langle [\lambda] (r' r'') | Q_1 | [\lambda] ((r'_1 r'_2) \lambda_1 r_2) \rangle \\ \times Q'_1 \varphi'(l_1^{n_1} [\lambda_1] (r'_1 r'_2) \alpha_1 L_1, l_2^{n_2} [\lambda_2] (r_2) \alpha_2 L_2; \\ \rightarrow LM | 1, 2, \dots, n_1-2, n-1, n; n_1-1, n_1, \dots, n-2). \end{aligned} \quad (6)$$

We strip off the particles $n-1$ and n from the first shell using the fractional parentage coefficients for that shell, and then change the order of coupling of the orbital angular momenta so that the group $l_1^{n_1-2}$ is coupled to $l_2^{n_2}$. When we do this, we get a transformation matrix for the rotation group, in other words, a matrix which transforms between different schemes for coupling the orbital angular momenta: [13]

$$\begin{aligned} \varphi' = \sum_{\alpha'_1 L'_1 L''_1 L'} \varphi''((l_1^{n_1-2} [\lambda'_1] (r'_1) \alpha'_1 L'_1, l_2^{n_2} [\lambda_2] (r_2) \alpha_2 L_2) L', \\ \times l_1^2 [\lambda''] L''_1; LM) \langle l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_1^2 [\lambda''] L''_1; L | \rangle l_1^{n_1} \\ \times [\lambda_1] \alpha_1 L_1 \rangle \langle ((L'_1 L_2) L' L'_1) L | ((L'_1 L'_2) L_1 L_2) L \rangle. \end{aligned} \quad (7)$$

We write the matrix element as $Q_1 = Q'_1 P_1$ according to the formula for multiplication of the matrices chosen for the intermediate state with permutation symmetry of the function φ'' and a

*We choose for Q the permutations which preserve the increasing order of numbering of particles within each shell.

definite permutation symmetry $[\bar{\lambda}']$ for the first $n-2$ numbers:

$$\begin{aligned} & \langle [\lambda] (r' r'') | Q_1 P_1 | [\lambda] ((\bar{r}'_1 \bar{r}'_2) \bar{\lambda}'_1 \bar{r}'_2) \rangle \\ &= \sum \langle [\lambda] (r' r'') | Q_1 | [\lambda] ((\bar{r}'_1 \bar{r}'_2) \bar{\lambda}'_1 \bar{r}'_2) \rangle \\ & \times \langle [\lambda] ((\bar{r}'_1 \bar{r}'_2) \bar{\lambda}'_1 \bar{r}'_2) | P_1 | [\lambda] ((\bar{r}'_1 \bar{r}'_2) \bar{\lambda}'_1 \bar{r}'_2) \rangle. \end{aligned} \quad (8)$$

Since $Q_1' \in S_{n-2}$, the sum over $\bar{\lambda}'$ is replaced by $\delta_{\bar{\lambda}' \bar{\lambda}'}$ and in addition we also get $\delta_{r'_1 r''}$. It is not difficult to show that P_1 is a matrix of type I (18), i.e. it is diagonal in $[\lambda_i] (r_i)$ and is independent of the Yamanouchi symbols. Consequently the summation in (8) drops out.

We substitute (7) and (8) in (6). Since

$$\begin{aligned} & \sum_{r'_1 r'_2} \sum_{Q_1'} \langle [\lambda'] (r') | Q_1' | [\lambda'] (r'_1 r'_2) \rangle Q_1' \Phi' ([\lambda'_1] (r'_1), [\lambda'_2] (r'_2), [\lambda'']) \\ &= c' \Phi' ([\lambda'_1] [\lambda'_2] | [\lambda'] (r'), [\lambda'']), \\ c' &= \left\{ \frac{f_{\lambda'_1} f_{\lambda'_2}}{f_{\lambda'}} \frac{(n-2)!}{(n_1-2)! n_2!} \right\}^{1/2}, \end{aligned} \quad (9)$$

the summation $\sum_1$ finally takes the following form*

$$\begin{aligned} \sum_1 &= c c' \sum_{\lambda'_1} \sum_{\lambda'_2} \Phi' ((l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda'_2] \alpha'_2 L'_2) | [\lambda'] (r') L', \\ & \times l_1^2 [\lambda''] L''; LM) \langle l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_1^2 [\lambda''] L''; L_1 \rangle l_1^{n_1} [\lambda_1] \\ & \times \alpha'_1 L'_1 \rangle \langle [\lambda] ((\lambda'_1 \lambda'_2) \lambda' \lambda'') \| P_1 \| [\lambda] ((\lambda'_1 \lambda'') \lambda_1 \lambda_2) \rangle \\ & \times \langle ((L'_1 L'_2) L' L'_1) L | ((L'_1 L'_2) L_1 L_2) L \rangle. \end{aligned} \quad (10)$$

Comparing this expression with the first term in (2), we get the following formula for the fractional parentage coefficients:

$$\begin{aligned} & \langle (l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda'_2] \alpha'_2 L'_2) | [\lambda'] L', l_1^2 [\lambda''] L''; L_1 \rangle l_1^{n_1} [\lambda_1] \\ & \times \alpha'_1 L'_1 \rightarrow l_2^{n_2} [\lambda_2] \alpha'_2 L'_2; [\lambda] L \rangle = \left\{ \frac{f_{\lambda} f_{\lambda'_1}}{f_{\lambda'} f_{\lambda_1}} \frac{n_1 (n_1-1)}{n (n-1)} \right\}^{1/2} \\ & \times \langle l_1^{n_1-2} [\lambda'_1] \alpha'_1 L'_1, l_1^2 [\lambda''] L''; L_1 \rangle l_1^{n_1} [\lambda_1] \alpha'_1 L'_1 \rangle \\ & \times \langle [\lambda] ((\lambda'_1 \lambda'_2) \lambda' \lambda'') \| P_1 \| [\lambda] ((\lambda'_1 \lambda'') \lambda_1 \lambda_2) \rangle \\ & \times \langle ((L'_1 L'_2) L' L'_1) L | ((L'_1 L'_2) L_1 L_2) L \rangle. \end{aligned} \quad (11)$$

Proceeding similarly with the fourth sum, which contains a summation over Q_4 , and comparing it with the second term in (2), we get

$$\begin{aligned} & \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2-2} [\lambda'_2] \alpha'_2 L'_2) | [\lambda'] L', l_2^2 [\lambda''] L''; L_2 \rangle \\ & \times l_1^{n_1} [\lambda_1] \alpha_1 L_1 \rightarrow l_2^{n_2} [\lambda_2] \alpha'_2 L'_2; [\lambda] L \rangle \\ &= \left\{ \frac{f_{\lambda} f_{\lambda'_2}}{f_{\lambda'} f_{\lambda_2}} \frac{n_2 (n_2-1)}{n (n-1)} \right\}^{1/2} \langle l_2^{n_2-2} [\lambda'_2] \alpha'_2 L'_2, l_2^2 [\lambda''] L''; L_2 \rangle \\ & \times l_2^{n_2} [\lambda_2] \alpha'_2 L'_2 \rangle \langle [\lambda] ((\lambda_1 \lambda'_2) \lambda' \lambda'') \| [\lambda] ((\lambda_1 \lambda'_2) \lambda_2) \rangle \\ & \times \langle ((L_1 L'_2) L' L'_2) L | (L_1 (L'_2 L'_2) L_2) L \rangle. \end{aligned} \quad (12)$$

*To simplify the writing, in indicating the method of reduction we write all the intermediate $[\lambda_i]$ without the square brackets.

To find the parentage coefficients $\langle l_1^{n_1-1} l_2^{n_2-1}, l_1 l_2 | l_1^{n_1} l_2^{n_2} \rangle$ we must combine the second and third sums. First we transform the third sum:

$$\begin{aligned} \sum_3 &= c \sum_{r'_1 r'_2} \sum_{Q'_2} \langle [\lambda] (r' r'') | P_{n, n-1} Q_2 | [\lambda] (r'_1 r'_2) \rangle Q_3 \Phi \\ &= c \sum_{r'_1 r'_2} \sum_{Q'_2} \langle [\lambda''] | P_{n, n-1} | [\lambda''] \rangle \langle [\lambda] (r' r'') | Q_2 | [\lambda] (r'_1 r'_2) \rangle Q_3 \Phi. \end{aligned} \quad (13)$$

Combining it with the second, we obtain

$$\begin{aligned} \sum_2 + \sum_3 &= c \sum_{r'_1 r'_2} \sum_{Q'_2} (1 + \langle [\lambda''] | P_{n, n-1} | [\lambda''] \rangle P_{n, n-1}) \\ & \times \langle [\lambda] (r' r'') | Q_2 | [\lambda] (r'_1 r'_2) \rangle Q_2 \Phi (l_1^{n_1} [\lambda_1] (r_1) \alpha_1 L_1, \\ & \rightarrow l_2^{n_2} [\lambda_2] (r_2) \alpha_2 L_2; LM | 1, 2 \dots n_1-1, n-1; n_1, \\ & n_1+1 \dots n-2, n). \end{aligned} \quad (14)$$

Next, separating off the particles $n-1$ and n from the first and second shells by using the single-shell parentage coefficients and carrying out some transformations, we get the required expression:

$$\begin{aligned} & \langle (l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2) | [\lambda'] L', l_1 l_2 [\lambda''] L''; L \rangle \\ & \times l_1^{n_1} [\lambda_1] \alpha_1 L_1 \rightarrow l_2^{n_2} [\lambda_2] \alpha_2 L_2; [\lambda] L \rangle \\ &= \left\{ \frac{f_{\lambda} f_{\lambda'_1} f_{\lambda'_2}}{f_{\lambda'} f_{\lambda_1} f_{\lambda_2}} \frac{2 n_1 n_2}{n (n-1)} \right\}^{1/2} \\ & \times \langle l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_1; L_1 \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \rangle \\ & \times \langle l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2, l_2; L_2 \rangle l_2^{n_2} [\lambda_2] \alpha_2 L_2 \rangle \\ & \times \langle [\lambda] ((\lambda'_1 \lambda'_2) \lambda' \lambda'') \| P_2 \| [\lambda] ((\lambda'_1 \lambda'_2) \lambda_1 (\lambda'_2 \lambda_2)) \rangle \\ & \times \langle ((L'_1 L'_2) L' L'') L | ((L'_1 L'_2) L_1 (L'_2 L'_2) L_2) L \rangle. \end{aligned} \quad (15)$$

To obtain the fractional parentage coefficients $\langle n-1, 1 | n \rangle$, we divide the sum over Q in I (20) into two sums. In the first sum we include the permutations which bring particle n to position n_1 . From it we obtain the form of the coefficients $\langle l_1^{n_1-1} l_2^{n_2}, l_1 | l_1^{n_1} l_2^{n_2} \rangle$. The second sum includes those permutations which leave the n -th particle fixed, and gives us $\langle l_1^{n_1} l_2^{n_2-1}, l_2 | l_1^{n_1} l_2^{n_2} \rangle$. As a result we have

$$\begin{aligned} & \langle (l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_2^{n_2} [\lambda'_2] \alpha'_2 L'_2) | [\lambda'] L', l_1; L \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1, \\ & \rightarrow l_2^{n_2} [\lambda_2] \alpha'_2 L'_2; [\lambda] L \rangle = \left\{ \frac{f_{\lambda} f_{\lambda'_1}}{f_{\lambda'} f_{\lambda_1}} \frac{n_1}{n} \right\}^{1/2} \langle l_1^{n_1-1} [\lambda'_1] \alpha'_1 L'_1, l_1; \\ & L_1 \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \rangle \langle [\lambda] ((\lambda'_1 \lambda'_2) \lambda' 1) \| P_{n, n-1 \dots n_1} \| [\lambda] \\ & \times ((\lambda'_1 1) \lambda_1 \lambda_2) \rangle \langle ((L'_1 L'_2) L' L'_1) L | ((L'_1 L'_2) L_1 L_2) L \rangle; \end{aligned} \quad (16)$$

$$\begin{aligned} & \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1, l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2) | [\lambda'] L', l_2; L \rangle l_1^{n_1} [\lambda_1] \alpha_1 L_1, \\ & \rightarrow l_2^{n_2} [\lambda_2] \alpha'_2 L'_2; [\lambda] L \rangle = \left\{ \frac{f_{\lambda} f_{\lambda'_2}}{f_{\lambda'} f_{\lambda_2}} \frac{n_2}{n} \right\}^{1/2} \langle l_2^{n_2-1} [\lambda'_2] \alpha'_2 L'_2; \\ & L_2 \rangle l_2^{n_2} [\lambda_2] \alpha_2 L_2 \rangle \langle [\lambda] ((\lambda_1 \lambda'_2) \lambda' 1) \| [\lambda] ((\lambda_1 \lambda'_2) \lambda_2) \rangle \\ & \times \langle ((L_1 L'_2) L' L'_2) L | (L_1 (L'_2 L'_2) L_2) L \rangle. \end{aligned} \quad (17)$$

All our formulas for the parentage coefficients are diagonal in the shells from which no particles were split off, and the transformation matrices repeat the coupling scheme in the two parts of the parentage coefficient. It is not hard to see that for the special case of the antisymmetric representation $[\lambda] = [11 \dots 1]$ the formulas for the parentage coefficients go over into the corresponding formulas in the paper of Levinson,^[8] since, for the one-dimensional representations the transformation matrices of the permutation group, which appear in the formulas, are equal to unity except for a phase factor.

2. ARBITRARY MULTISHELL CONFIGURATION

When there are k shells present, to characterize a state with a given permutation symmetry and total orbital angular momentum we must also give $k-2$ intermediate symmetry patterns and $k-2$ intermediate orbital angular momenta. We shall define the fractional parentage coefficients for such a configuration by generalizing the expansion of the coordinate wave function for two shells given in (1) and (2):

$$\begin{aligned} & \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^B b_\lambda [\lambda] (r) b_L L M) \\ &= \sum_{i=1}^k \sum_{\lambda_i' b_{\lambda_i}'} \sum_{\alpha_i' L_i'} \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' \dots \\ & \rightarrow l_i^{n_i} [\lambda_k] \alpha_k L_k)^{B'} b_{\lambda'} [\lambda'] (r') b_{L', L'} l_i; LM) \\ & \times \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' \dots l_k^{n_k} [\lambda_k] \\ & \times \alpha_k L_k)^{B'} b_{\lambda'} [\lambda'] b_{L', L'} \\ & \rightarrow l_i; L \rangle \langle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots \rangle^B b_\lambda [\lambda] b_L L \rangle. \end{aligned} \quad (18)$$

B and B' are the coupling schemes for the shell symmetry patterns and orbital angular momenta (cf. I); b_λ and b_L are respectively the sets of intermediate symmetry patterns and intermediate orbital angular momenta. The summation extends only over those symmetry patterns and orbital angular momenta in $b_{\lambda'}$ and $b_{L'}$ which appear when we strip off a particle. Furthermore,

$$\begin{aligned} & \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^B b_\lambda [\lambda] (r' r'') b_L L M) \\ &= \sum_{i=1}^k \sum_{\lambda_i' b_{\lambda_i}'} \sum_{\alpha_i' L_i'} \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-2} [\lambda_i'] \alpha_i' L_i' \dots \\ & \times \alpha_i' L_i' \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^{B'} \rightarrow b_{\lambda'} [\lambda'] (r') b_{L', L'} l_i^2 [\lambda''] \\ & \times L_i''; LM) \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-2} [\lambda_i'] \alpha_i' L_i' \dots l_k^{n_k} [\lambda_k] \\ & \times \alpha_k L_k)^{B'} b_{\lambda'} [\lambda'] b_{L', L'} \rightarrow l_i^2 [\lambda''] L_i''; L \rangle \\ & \times \langle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots \rangle^B b_\lambda [\lambda] b_L L \rangle \end{aligned}$$

$$\begin{aligned} & + \sum_{i < j} \sum_{\lambda_i' b_{\lambda_i}'} \sum_{\alpha_i' L_i'} \Phi((l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' \dots \\ & \times \alpha_i' L_i' \dots \rightarrow l_j^{n_j-1} [\lambda_j'] \alpha_j' L_j' \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^{B'} b_{\lambda'} [\lambda'] \\ & \times (r') b_{L', L'} l_i l_j [\lambda''] L''; LM) \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' \dots \\ & \times \alpha_i' L_i' \dots l_j^{n_j-1} [\lambda_j'] \alpha_j' L_j' \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^{B'} \\ & \rightarrow b_{\lambda'} [\lambda'] b_{L', L'} l_i l_j [\lambda''] L''; L \rangle \langle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots \rangle^B b_\lambda [\lambda] \\ & \times [\lambda] b_L L \rangle. \end{aligned} \quad (19)$$

To obtain the fractional parentage coefficients defined in (18) and (19) we shall start from the expression I (21) for the wave function. The computations are similar to those for the case of two shells, but are more complicated. We shall give only the results.

$$\begin{aligned} & \text{In the case of } \langle n-1, 1 | \rangle n, \\ & \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^B b_{\lambda'} [\lambda'] b_{L', L'} \\ & \rightarrow l_i; L \rangle \langle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots \rangle^B b_\lambda [\lambda] b_L L \rangle \\ &= \left\{ \frac{f_{\lambda} f_{\lambda_i'}}{f_{\lambda'} f_{\lambda_i}} \frac{n_i}{n} \right\}^{1/2} \langle l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' l_i; L \rangle \langle l_i^{n_i} [\lambda_i] \alpha_i L_i \rangle \\ & \times \langle [\lambda] ((\lambda_1 \dots \lambda_i' \dots \lambda_k)^{B'} b_{\lambda'} \lambda' 1) \| P_i^{(1)} \| [\lambda] ((\lambda_1 \dots (\lambda_i' 1) \\ & \times \lambda_i \dots \lambda_k)^B b_\lambda) \rangle \langle (L_1 \dots L_i' \dots L_k)^{B'} b_{L', L'} L_i \rangle \\ & \times L | (L_1 \dots (L_i' L_i) L_i \dots L_k)^B b_L L \rangle. \end{aligned} \quad (20)$$

In the case of $\langle n-2, 2 | \rangle n$, for the stripping off of two particles from the same shell,

$$\begin{aligned} & \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-2} [\lambda_i'] \alpha_i' L_i' \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^{B'} b_{\lambda'} [\lambda'] \\ & \times b_{L', L'} \rightarrow l_i^2 [\lambda''] L_i''; L \rangle \langle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots \rangle^B b_\lambda [\lambda] b_L L \rangle \\ &= \left\{ \frac{f_{\lambda} f_{\lambda_i'}}{f_{\lambda'} f_{\lambda_i}} \frac{n_i (n_i-1)}{n (n-1)} \right\}^{1/2} \langle l_i^{n_i-2} [\lambda_i'] \alpha_i' L_i' l_i^2 [\lambda''] L_i''; L \rangle \langle l_i^{n_i} [\lambda_i] \alpha_i L_i \rangle \\ & \times | [\lambda_i] \alpha_i L_i \rangle \langle [\lambda] ((\lambda_1 \dots \lambda_i' \dots \lambda_k)^{B'} b_{\lambda'} \lambda' \lambda'') \| P_i^{(2)} \| [\lambda] \\ & \times ((\lambda_1 \dots (\lambda_i' \lambda'') \lambda_i \dots \lambda_k)^B b_\lambda) \rangle \langle (L_1 \dots L_i' \dots L_k)^{B'} \\ & \times b_{L', L'} L_i' \rangle L | (L_1 \dots (L_i' L_i) L_i \dots L_k)^B b_L L \rangle. \end{aligned} \quad (21)$$

For stripping off from different shells,

$$\begin{aligned} & \langle (l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' \dots l_j^{n_j-1} [\lambda_j'] \alpha_j' L_j' \dots \\ & \dots l_k^{n_k} [\lambda_k] \alpha_k L_k)^{B'} b_{\lambda'} [\lambda'] b_{L', L'} \rightarrow l_i l_j [\lambda''] L''; L \rangle \langle l_1^{n_1} [\lambda_1] \alpha_1 L_1 \dots \rangle^B b_\lambda [\lambda] b_L L \rangle \\ &= \left\{ \frac{f_{\lambda} f_{\lambda_i'} f_{\lambda_j'}}{f_{\lambda'} f_{\lambda_i} f_{\lambda_j}} \frac{2 n_i n_j}{n (n-1)} \right\}^{1/2} \\ & \times \langle l_i^{n_i-1} [\lambda_i'] \alpha_i' L_i' l_i; L \rangle \langle l_i^{n_i} [\lambda_i] \alpha_i L_i \rangle \langle l_j^{n_j-1} [\lambda_j'] \alpha_j' L_j' l_j; \\ & \times L_j \rangle \langle l_j^{n_j} [\lambda_j] \alpha_j L_j \rangle \langle [\lambda] ((\lambda_1 \dots \lambda_i' \dots \lambda_j' \dots \lambda_k)^{B'} b_{\lambda'} \lambda' \lambda'') \| P_{ij}^{(2)} \| \\ & \times [\lambda] ((\lambda_1 \dots (\lambda_i' 1) \lambda_i \dots (\lambda_j' 1) \lambda_j \dots \lambda_k)^B b_\lambda) \rangle \\ & \times | \langle (L_1 \dots L_i' \dots L_j' \dots L_k)^{B'} b_{L', L'} L' \rangle L | (L_1 \dots (L_i' L_i) L_i \dots (L_j' L_j) \\ & \times L_j \dots L_k)^B b_L L \rangle. \end{aligned} \quad (22)$$

All the permutations in formulas (20)–(22) preserve the increasing order of numbering of par-

ticles within the shells. $P_i^{(1)}$ moves n to the position of the last particle in the i -th shell, whose number we denote by $m_i = \sum_{t=1}^i n_t$; $P_i^{(2)}$ shifts n and $n-1$ into the i -th shell; $P_{ij}^{(2)}$ takes particle $n-1$ into the i -th shell and particle n into the j -th shell, $m_i < m_j$:

$$P_i^{(1)} = P_{n, n-1 \dots m_i};$$

$$P_i^{(2)} = \begin{cases} P_{n, n-2 \dots m_i-1, n-1, n-3 \dots m_i} & (n-m_i) \text{ odd,} \\ P_{n, n-2 \dots m_i} P_{n-1, n-3 \dots m_i-1} & (n-m_i) \text{ even,} \end{cases}$$

$$P_{ij}^{(2)} = \begin{cases} P_{n, n-2 \dots m_j-1, m_j-2 \dots m_i, n-1, n-3 \dots m_j} & (n-m_j) \text{ odd,} \\ P_{n, n-2 \dots m_j} P_{n-1, n-3 \dots m_j-1, m_j-2 \dots m_i} & (n-m_j) \text{ even.} \end{cases} \quad (23)$$

3. COMPUTATION OF FRACTIONAL PARENTAGE COEFFICIENTS

The expressions found for the fractional parentage coefficients consist of four factors: 1) a normalization factor, 2) parentage coefficients for a single shell, 3) matrix elements of transformation matrices of the permutation group, and 4) matrix elements of transformation matrices of the rotation group. $[\lambda] \equiv [\lambda^{(1)} \lambda^{(2)} \dots \lambda^{(m)}]$ of the permutation group S_n is given by the formula

$$f_\lambda = n! \prod_{i < j}^m (h_i - h_j) / h_1! \dots h_m!, \quad h_i = \lambda^{(i)} + m - i. \quad (24)$$

Single-shell parentage coefficients for stripping off of one particle for the d and p shells are given in^[2] and^[3]; those for stripping off two particles for the p shell are in^[5]. Coefficients for stripping off two particles can be found from those for stripping off one particle by using the formulas of^[5]. The transformation matrices for the rotation group are studied in detail in the monograph of Yutsis, Levinson, and Vanagas,^[13] who also give a complete bibliography of available tables of these matrices.

The transformation matrices of the permutation group can be calculated in terms of the matrices of the standard Young-Yamanouchi^[3,15] representation by using the method developed in I.* Let us consider some examples:

$$\langle [\lambda] ((\lambda'_1 \lambda'_2) \lambda' \lambda'') \| P_1 \| [\lambda] ((\lambda'_1 \lambda'_2) \lambda_1 \lambda_2) \rangle = \sum_{r, \bar{r}} \langle [\lambda] ((r'_1 r'_2) \lambda' r'') \times [\lambda] (r) \rangle \langle [\lambda] (r) | P_1 | [\lambda] (\bar{r}) \rangle \langle [\lambda] (\bar{r}) | [\lambda] ((r'_1 r'_2) \lambda_1 \lambda_2) \rangle. \quad (25)$$

*Calculations are being made at present of tables of transformation matrices of the permutation group which appear in the formulas for fractional parentage coefficients of two-shell configurations with $N = 3-6$. Tables are also being prepared of coordinate parentage coefficients for vector-uncoupled states with numbers of particles from three to six.

The Yamanouchi symbols in the matrices for transition from the standard basis to the non-standard one can be chosen in any convenient manner, since the final expression does not depend on them. The two transformation matrices are found from formulas I (13) and I (8). Conditions like I (10):

$$\{\lambda_1 (\lambda'_1 \lambda'')\}, \quad \{\lambda (\lambda_1 \lambda_2)\}, \quad \{\lambda' (\lambda'_1 \lambda'_2)\}, \quad \{\lambda (\lambda' \lambda'')\} \quad (26)$$

are imposed on the symmetry patterns in (25). We find the matrix appearing in (12) by choosing as the intermediate state the one with the standard pattern in the reduction:

$$\langle [\lambda] ((\lambda'_1 \lambda'_2) \lambda' \lambda'') | [\lambda] (\lambda_1 (\lambda'_2 \lambda'') \lambda_2) \rangle = \sum_{(r)} \langle [\lambda] ((r'_1 r'_2) \lambda' r'') | [\lambda] (r) \rangle \langle [\lambda] (r) | [\lambda] (r_1 (r'_2 r'') \lambda_2) \rangle. \quad (27)$$

Similarly we find the matrices with more complicated reduction types.

4. COMPUTATION OF MATRIX ELEMENTS

In those cases where the operators do not depend on the spin coordinates, a knowledge of the coordinate wave functions is sufficient for computing the matrix elements. We start from the expression for the antisymmetric total wave function in the form of a sum of products of coordinate and spin functions,* symmetrized according to associated representations:^[3]

$$\Psi = \frac{1}{\sqrt{f_\lambda}} \sum_{(r)} \Phi_{(r)}^{[\lambda]} \chi_{(\bar{r})}^{[\bar{\lambda}]}. \quad (28)$$

Using the orthonormality of the spin functions and the expansions (18) and (19), we can express the matrix elements of the operators F and G for an arbitrary multishell configuration in terms of the fractional parentage coefficients and matrix elements for one or two particles. Here, because of the identity of the particles or, what amounts to the same thing, because of the antisymmetry of the total wave function, the operator F can be replaced by $n f_n$, and G by $\frac{1}{2} n(n-1) g_{n-1, n}$. The operator f_n acts only on the coordinates of the last (n -th) particle, and $g_{n-1, n}$ on the coordinates of the last two particles; we shall assume it to be a scalar, and denote it simply by g .

For the matrix elements of the operator F we get

*For nucleons we have a spin-charge function in place of the spin function, so in the following formulas we must write the isotopic spin T in addition to the spin S in the matrix elements.

$$\begin{aligned}
 & \langle \alpha [\lambda] L M_L \beta S M_S | F | \bar{\alpha} [\lambda] \bar{L} \bar{M}_L \bar{\beta} \bar{S} \bar{M}_{\bar{S}} \rangle \\
 &= \delta (\beta S M_S, \bar{\beta} \bar{S} \bar{M}_{\bar{S}}) n \sum_{i=1}^k \sum_{\alpha' \lambda' L'} \frac{f_{\lambda'}}{f_{\lambda}} \langle \alpha [\lambda] L \{ | \alpha' [\lambda'] \\
 & \times | L', l_i; L \rangle \langle \alpha' [\lambda'] L', l_i; L M | f_n | \alpha' [\lambda'] L', l_i; \bar{L} \bar{M} \rangle \\
 & \times \langle \alpha' [\lambda'] L', l_i; \bar{L} \} \bar{\alpha} [\lambda] \bar{L} \rangle. \quad (29)
 \end{aligned}$$

To shorten the writing we have omitted the designation of the configurations; the sets of orbital quantum numbers which distinguish terms of the same type are denoted by α , and the set of spin quantum numbers by β . The matrix elements of the operator f_n can be determined by using the algebra of tensor operators developed by Racah.^[16] For the scalar operator G we have

$$\begin{aligned}
 \langle \alpha [\lambda] L \beta S | G | \bar{\alpha} [\lambda] L \beta S \rangle &= \frac{n(n-1)}{2} \sum_{i < j}^k \sum_{\alpha' \lambda' L' \lambda'' L''} \frac{f_{\lambda'}}{f_{\lambda}} \\
 & \times \langle \alpha [\lambda] L \{ | \alpha' [\lambda'] L', l_i l_j [\lambda''] L''; L \rangle \\
 & \times \langle l_i l_j [\lambda''] L'' | g | l_i l_j [\lambda''] L'' \rangle \\
 & \times \langle \alpha' [\lambda'] L', l_i l_j [\lambda''] L''; L | \bar{\alpha} [\lambda] L \rangle. \quad (30)
 \end{aligned}$$

A detailed description of the computation of orbital matrix elements of the operator g can be found, for example, in the paper of Elliott and Lane (cf.^[17]). In the matrix element formulas (29) and (30) it is assumed that the same configuration appears on the right and left hand sides. In a similar way one can obtain formulas for the nonvanishing matrix elements with configurations which differ by having one or two particles permuted between the shells.

In conclusion, I thank I. B. Levinson for much valuable advice during the course of the work, and A. S. Kompaneets and V. G. Neudachin for discussion of the results.

¹G. Racah, Phys. Rev. **63**, 367 (1943); **76**, 1352 (1949).

²H. A. Jahn, Proc. Roy. Soc. (London) **A205**, 192 (1951).

³H. A. Jahn and H. van Wieringen, Proc. Roy. Soc. (London) **A209**, 502 (1951).

⁴A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952).

⁵Elliott, Hope and Jahn, Phil. Trans. Roy. Soc. **A246**, 241 (1953).

⁶P. J. Redmond, Proc. Roy. Soc. (London) **A222**, 84 (1954).

⁷A. Hassitt, Proc. Roy. Soc. (London) **A229**, 110 (1955).

⁸I. B. Levinson, Reports of the Academy of Sciences, Lithuanian SSR, **B4**, 17 (1957).

⁹J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

¹⁰Balashov, Tumanov and Shirokov, Nuclear Reactions at Low and Medium Energies, Reports of the All-Union Conference, 1957-58, p. 548.

¹¹E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra, Cambridge, University Press, 1935.

¹²I. G. Kaplan, JETP **41**, 560 (1961). Soviet Phys. JETP **14**, 401 (1962).

¹³Yutsis, Levinson and Vanagas, Matematicheskii Apparat Teori Momenta Kolichestva Dvizheniya (Mathematical Apparatus of the Theory of Angular Momentum) Vilna, 1960.

¹⁴F. Murnaghan, The Theory of Group Representations, The Johns Hopkins Press, Baltimore, 1938.

¹⁵D. E. Rutherford, Substitutional Analysis, Edinburgh University Press, 1948.

¹⁶G. Racah, Phys. Rev. **62**, 438 (1942). G. Racah, Group Theory and Spectroscopy, mimeographed lecture notes, Princeton, 1951.

¹⁷J. P. Elliott and A. M. Lane, Handbuch der Physik, Vol. 39, 1957.

Translated by M. Hamermesh

INTERACTION WITH THE SURFACE OF NUCLEI CONTAINING A NUCLEON IN EXCESS OF A CLOSED SHELL

V. N. GUMAN

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor March 16, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 800-805 (September, 1961)

Nuclei with doubly filled shells with one added nucleon or with one nucleon missing are investigated. Single-particle nucleon energies due only to the central part of the self-consistent potential are singled out by taking into account the interaction with the nuclear surface vibrations. Surface interaction parameters are derived, which explain the experimentally established quadrupole moments and the E2 transition probabilities. The applicability of the perturbation method to light nuclei is discussed.

THE shell potential in which nucleons move in the nucleus is usually determined by investigating nuclei having doubly-filled shells with one nucleon removed or added.^[1,2] Thus, for example, Sliv and Volchok^[1] have determined the parameters of the potential in the Woods-Saxon form. They noted that the nucleon is acted on not only by the central part of the potential but by an additional part connected with the deformation of the potential surface. However, a hypothesis has been advanced that account of this interaction results in a small level shift (on the order of 100 kev), which does not influence the choice of the parameters and whose influence on the E2 transitions can be determined by perturbation theory.

A check of these assumptions for O^{17} , Pb^{207} , Pb^{209} , and Bi^{209} , reported in the present paper, has shown that the interaction due to the deformation of the potential surface is not small in the case of heavy nuclei, and is particularly large for light nuclei. A more accurate calculation shows that perturbation theory cannot be used to account for the quadrupole moments and for the transition probabilities with the determined parameters of the interaction with the surface, not only in the region of light nuclei, but also for certain heavy nuclei (Pb^{207}). We have obtained the true single-particle nucleon excitation energies, due only to the central potential of the self-consistent nuclear field.

THE NUCLEUS O^{17}

The nucleus O^{17} was chosen for research in the region of light nuclei, since it has been relatively thoroughly investigated; the quadrupole moment

and probability of E2 transition ($1/2 \rightarrow 5/2$) have been measured for this nucleus.

The motion of the external neutron and of the surface of the nucleus are described by a Hamiltonian

$$\hat{H} = \hat{H}_C(r) + V_S(\alpha, r) + \hat{H}_S(\alpha). \quad (1)$$

Here $H_C(r)$ is the central part of the self-consistent potential^[1]:

$$H_C = \frac{V^2}{2M} + V(r) - \lambda \left(\frac{\hbar}{2Mc} \right)^2 \frac{1}{r} \frac{dV}{dr}, \quad (2)$$

where $V(r)$ is the Woods-Saxon potential

$$V(r) = V_0 / (1 + e^{\alpha(r-r_0)}); \quad (3)$$

$V_S(\alpha, r)$ is the quadrupole terms of the expansion of the self-consistent potential in Legendre polynomials.

V_S can be regarded as an interaction between the nucleon and the oscillations of the potential surface:

$$V_S(\alpha, r) = -k(r) \sum_{\mu} \hat{\alpha}_{2\mu} Y_{2\mu}(\theta, \varphi), \quad (4)$$

where $\alpha_{2\mu}$ are the parameters of the nuclear deformation^[3]:

$$\alpha_{2\mu} = \sqrt{\hbar\omega/2C} (b_{\mu} + (-1)^{\mu} b_{-\mu}^*), \quad (5)$$

b_{μ}^* and b_{μ} are the creation and annihilation operators for vibrational phonons with spin 2. Then the Hamiltonian H_S , describing the collective oscillations of the surface, is

$$H_S = \hbar\omega \sum_{\mu} (1/2 + b_{\mu}^* b_{\mu}). \quad (6)$$

For small deformations of the potential surface we have

$$k(r) = r \partial V / \partial r. \quad (7)$$

Calculation shows that independently of the states of the particles, the matrix elements $(n'l | k(r) | n'l')$ are approximately constant and have a value

$$(n'l | k(r) | n'l') = \int_0^\infty R_{nl}(r) R_{n'l'}(r) r^3 \frac{\partial V}{\partial r} dr \approx (-1)^{n+n'} k, \quad (8)$$

where $R_{nl}(r)$ are the radial functions in the potential (2). For heavy nuclei in the lead region $k \approx 36-40$ Mev, while for O^{17} we have $k \approx 30-35$ Mev. If we assume $k = 40$ Mev, as is customarily done, the constant C in (5) assumes the meaning of the effective surface tension.

The eigenfunctions and the eigenvalues of the operators $H_S(\alpha)$ and $H_C(r)$ are known. An eigenfunction of the Hamiltonian (1) is a superposition of the product of the eigenfunctions of Hamiltonians (2) and (4), with different particle angular momentum j and angular momentum of the surface R ; only the total momentum of the level I is conserved, with

$$\psi_I = \sum_{jNR} c'_{jNR} |j; NR; I\rangle. \quad (9)$$

The term V_S of (4) is determined by the diagonalization of the energy matrices in the three-phonon approximation. The nuclear energy levels and the coefficients c'_{jNR} will be functions of two parameters, namely C and the phonon energy $\hbar\omega$. The matrix element of the operator V_S is

$$\begin{aligned} (j; NR; I | -k \sum_{\mu} \alpha_{2\mu} Y_{2\mu} | j'; N'R'; I) \\ = -(-1)^{n+n'} \cdot 40 \sqrt{\hbar\omega/2C} \delta_{N,N'+1} (-1)^{I+R'-I} \\ \times (j \| Y^2 \| j') (NR \| b^2 \| N'R') W(jRj'R'; I2). \end{aligned} \quad (10)$$

The matrix elements $(NR \| b^2 \| N'R')$ given here are taken from the paper by Raz^[4] and $W(jRj'R'; I2)$ are Racah coefficients.

Tables I and II list the coefficients c'_{jNR} for $C = 100$ Mev with $\hbar\omega$ ranging from 1 to 10 Mev. When $\hbar\omega$ is 1 and 2 Mev, the first excited level with $I = \frac{1}{2}$ is the level with a single phonon; but when $\hbar\omega$ has values from 3 to 10 Mev, the admixture of levels with phonons is sufficiently large. Since the coefficients of mixture c'_{jNR} depend relatively weakly on the choice of the single-particle level energy $E_{2S1/2}$, we can calculate the quadrupole moment and the reduced probability of the $E2$ transition $\frac{1}{2} \rightarrow \frac{5}{2}$ without specifically fixing this energy. The quadrupole moment of the external neutron is determined by the collective part of the operator Q ^[3]:

$$Q_S = 3(5\pi)^{-1/2} Z R_0^2 \langle \alpha_{20} \rangle_{M=I}. \quad (11)$$

The matrix element of the operator α_{20} has the form

$$\begin{aligned} (j; NR; II | \alpha_{20} | j'; N'R'; II) = C_{II20}^I \delta_{j'j} (-1)^{I-R-I} \sqrt{\hbar\omega/2C} \\ \times (NR \| b^2 \| N'R') \sqrt{2I+1} W(RIR' I; j2). \end{aligned} \quad (12)$$

Let us calculate now the transition probability. By reduced transition probability we mean

$$B(E2) = T(E2) / (\Delta E)^5, \quad (13)$$

where $T(E2)$ is the total probability of γ transition per second; ΔE is measured in Mev. Then

$$B_{\text{exp}}(E2) = 0.79 \cdot 10^{10} \text{sec}^{-1} \cdot \text{Mev}^{-5}.$$

Since diagonalization causes the wave function of state (9) to be a mixture of states with different j and different number of phonons, $B_{E2}(I \rightarrow I')$ has the form

$$\begin{aligned} B(E2) = \frac{4\pi}{75 \hbar (\hbar c)^5} \left[\frac{3ZeR_0^2}{4\pi} \right]^2 \frac{\hbar\omega}{C} (2I' + 1) \\ \times \left\{ \sum_{jNR, j'N'R'} c'_{jNR} c'_{j'N'R'} (-1)^{I-I'} (NR \| b^2 \| N'R') \right. \\ \left. \times W(RIR'I'; j2) \right\}^2, \end{aligned} \quad (14)$$

Table I. Mixture coefficients c'_{jNR} for $C = 100$ Mev at various values of $\hbar\omega$ for the level $I = \frac{5}{2}$

	$\hbar\omega$, Mev					
	1	2	3	5	7	10
$d_{5/2}; 00; \frac{5}{2}$	0.762	0.830	0.866	0.906	0.926	0.945
$d_{5/2}; 12; \frac{5}{2}$	-0.524	-0.443	-0.392	-0.327	-0.287	-0.248
$d_{5/2}; 20; \frac{5}{2}$	0.168	0.116	0.090	0.062	0.048	0.035
$d_{5/2}; 22; \frac{5}{2}$	0.029	0.033	0.031	0.025	0.021	0.016
$d_{5/2}; 24; \frac{5}{2}$	0.186	0.140	0.113	0.082	0.064	0.048
$d_{5/2}; 30; \frac{5}{2}$	-0.025	-0.017	-0.012	-0.007	-0.005	-0.003
$d_{5/2}; 32; \frac{5}{2}$	-0.044	-0.023	-0.015	-0.008	-0.006	-0.003
$d_{5/2}; 34; \frac{5}{2}$	-0.012	-0.009	-0.007	-0.004	-0.003	-0.002
$s_{1/2}; 12; \frac{5}{2}$	-0.243	-0.253	-0.255	-0.240	-0.222	-0.201
$s_{1/2}; 22; \frac{5}{2}$	0.124	0.100	0.081	0.059	0.046	0.035
$s_{1/2}; 32; \frac{5}{2}$	-0.046	-0.031	-0.022	-0.013	-0.009	-0.006

Table II. Mixture coefficients c_{jNR}^I for $C = 100$ Mev
at various values of $\hbar\omega$ for the level $I = 1/2$

	$\hbar\omega$, MeV					
	1	2	3	5	7	10
$d_{5/2}; 12; 1/2$	0.762	0.719	-0.661	-0.564	-0.495	-0.423
$d_{5/2}; 22; 1/2$	-0.385	-0.276	0.211	0.140	0.103	0.074
$d_{5/2}; 32; 1/2$	0.158	0.097	-0.065	-0.036	-0.023	-0.014
$s_{1/2}; 00; 1/2$	-0.442	-0.603	0.701	0.805	0.859	0.901
$s_{1/2}; 20; 1/2$	-0.210	-0.176	0.143	0.101	0.077	0.056
$s_{1/2}; 30; 1/2$	0.076	0.050	-0.035	-0.019	-0.013	-0.008

where

$$R_0 = 1.2 A^{1/3} \cdot 10^{-13} \text{ cm}.$$

As can be seen from Figs. 1 and 2, the quadrupole moment Q and particularly the reduced transition probability $B(E2)$ are very sensitive to changes in C and depend little on $\hbar\omega$. We can state immediately that the values 50 and 300 Mev are not suitable for C , and that C ranges from 100 to 200 Mev. The value of $\hbar\omega$ is determined quite stringently by the probability of transition from the second excited level. But this probability has not been measured experimentally, and we can therefore use for the choice of $\hbar\omega$ the calculations of the energy levels and transition probabilities for the neighboring nucleus O^{18} , for which $\hbar\omega = 2-5$ Mev. It is seen from Figs. 1-4 that when $\hbar\omega < 7$ Mev perturbation theory, according to which Q and $B(E2)$ are independent of $\hbar\omega$,^[5] is inapplicable. Choosing $\hbar\omega \approx 4$ Mev with $C \approx 130$ Mev, we find that Q and $B(E2)$ are close to the experimental values:

$$Q_{\text{exp}} = -0.26 \cdot 10^{-25} \text{ cm}^2,$$

$$B(E2)_{\text{exp}} = 0.79 \cdot 10^{10} \text{ sec}^{-1} \text{ Mev}^{-5}.$$

Calculations yield

$$Q(\hbar\omega = 3; C = 120) = -0.3 \cdot 10^{-25} \text{ cm}^2,$$

$$B(E2)(\hbar\omega = 3; C = 120) = 0.65 \cdot 10^{10} \text{ sec}^{-1} \text{ Mev}^{-5},$$

$$Q(\hbar\omega = 4; C = 130) = -0.29 \cdot 10^{-25} \text{ cm}^2,$$

$$B(E2)(\hbar\omega = 4; C = 130) = 0.70 \cdot 10^{10} \text{ sec}^{-1} \text{ Mev}^{-5}.$$

Having determined $\hbar\omega$ and C , we must take account of the level shifts due to the interaction with the surface. Whereas in perturbation theory these shifts are independent of the state in the second approximation, a more accurate calculation shows that the interaction is much stronger for $I = 1/2$ than for $I = 5/2$. Thus, for $C = 100$ Mev, the level $I = 1/2$ drops by almost 2.5 Mev, while the level $I = 5/2$ drops by 1 Mev (see Fig. 3). Therefore, when the interaction with the surface is taken into account, these levels are closer together than in the single-particle case. Thus, the experimentally measured position of the level $I = 1/2$ is much lower than its single-particle value. With $\hbar\omega = 4$ Mev and $C = 130$ Mev, the distance between levels agrees with the experimental value (0.87 Mev) at a single-particle energy $E_{S1/2} = 1.8$ Mev.

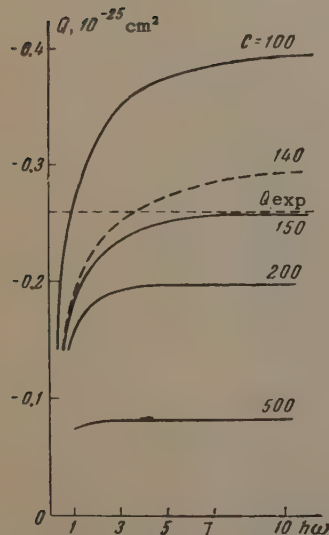


FIG. 1. Quadrupole moment of O^{17} for various C as a function of $\hbar\omega$.

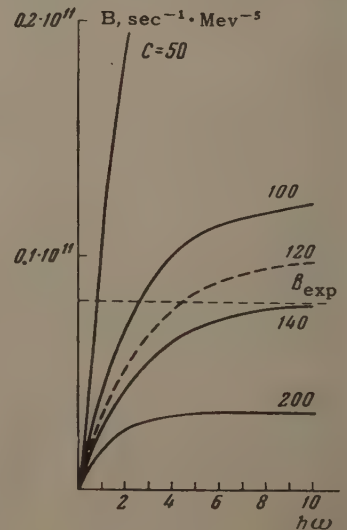


FIG. 2. Reduced probability of E2 transition ($1/2 \rightarrow 5/2$) in O^{17} for various C as a function of $\hbar\omega$.

Table III. Comparison of calculated and experimental data*
for $\hbar\omega = 3$ Mev and $C = 1000$ Mev

Nucleus	E, Mev	E_{exp} , Mev	Q, 10^{-24} cm ²		T(E2), 10^{10} sec ⁻¹	
			theory	expt.	theory	expt.
Pb ²⁰⁷	0.48 ($I = 5/2$)	0.57	—	—	0.9	0.9
Pb ²⁰⁹	0.76 ($I = 11/2$)	0.75	-0.26	—	—	—
Bi ²⁰⁹	0.93 ($I = 7/2$)	0.91	-0.46	-0.4	0.6	—

*E — calculated single-particle energy, E_{exp} — experimental location of excited levels.

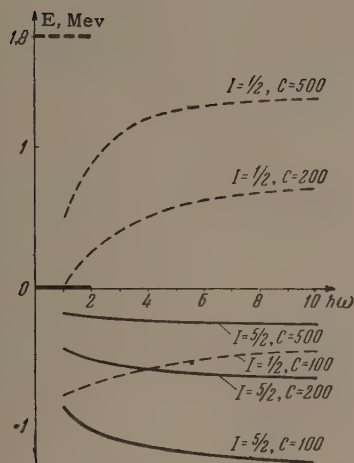


FIG. 3. Change in energy levels of O¹⁷ due to interaction with the surface, as a function of $\hbar\omega$. The single-particle energies are shown by the heavy lines (levels with $I = 5/2$ are shown solid, levels with $I = 1/2$ dotted).

THE NUCLEI Pb²⁰⁷, Pb²⁰⁹, AND Bi²⁰⁹

The nuclei Pb²⁰⁷, Pb²⁰⁹, and Bi²⁰⁹ were investigated analogously. The parameters of the surface interaction were assumed to be approximately the same for these nuclei, inasmuch as the nuclei are close.

From the experimentally known quadrupole moment of Bi²⁰⁹ and of the probability of the E2 transition in Pb²⁰⁷ ($5/2 \rightarrow 1/2$) we can determine $\hbar\omega$ and C.

Table III lists the calculated data, from which it is seen that when $\hbar\omega = 3$ Mev and $C = 1000$ Mev,

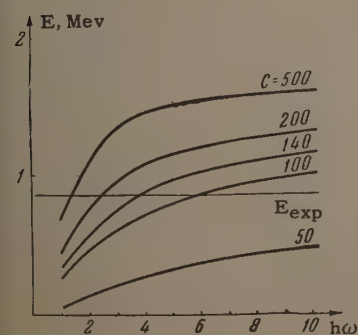


FIG. 4. Location of level $I = 1/2$ in O¹⁷ as a function of $\hbar\omega$ with $E_{2s1/2} = 1.8$ Mev.

the calculated values of Q and T(E2) are close to the experimental ones. If we take $C = 2000$ Mev, then the quadrupole moment of Bi²⁰⁹ is also close to the experimental value $Q = 0.35 \times 10^{10}$ sec⁻¹. It is obvious that for Pb²⁰⁹ and Bi²⁰⁹ the value of C lies between 1000 and 2000 Mev. The transition probability in Pb²⁰⁷ with $C = 200$ Mev is found to be too low, $T(E2) = 0.25 \times 10^{10}$ sec⁻¹; it is consequently better to take $C = 1000$ Mev for Pb²⁰⁷.

Choosing the interaction parameters in this fashion, we can calculate the level shifts in these nuclei. The interaction is most appreciable for Pb²⁰⁷ (see Fig. 5). For Bi²⁰⁹ and Pb²⁰⁹, the distance between levels remains practically unchanged. In the region of heavy nuclei, calculation by perturbation theory does not introduce appreciable errors in the quadrupole moment and in the transition probability from the first level. For example, assuming $C = 1000$ Mev, we obtain in the second perturbation-theory approximation $T(E2) = 0.93 \times 10^{10}$ sec⁻¹ for Pb²⁰⁷ and $Q = -0.50 \times 10^{-24}$ cm² for Bi²⁰⁹. When $C = 2000$ Mev, $Q = -0.35 \times 10^{-24}$ cm² for Bi²⁰⁹.

However, if the probability of transition from the next levels is taken into account, the results

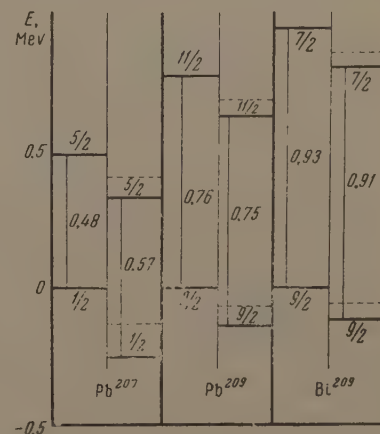


FIG. 5. Account of the interaction with the surface for heavy nuclei. Left column — single particle energies, right column — level shift due to interaction for $\hbar\omega = 3$ Mev, $C = 1000$ Mev; the dashed lines show the levels shifts for $\hbar\omega = 3$ Mev and $C = 2000$ Mev.

of perturbation theory are no longer correct,^[6] for they can differ by several orders of magnitude from those calculated by the diagonalization method. For levels with small j (as for example in Pb^{207}) the determination of the correct level shifts calls also for a more accurate account of the interaction with the surface.

The author considers it his pleasant duty to express deep gratitude to L. A. Sliv, under whose guidance this work was carried out, and also to G. A. Sogomonova, who participated in the calculations.

¹ L. A. Sliv and B. A. Volchok, JETP **36**, 539 (1959), Soviet Phys. JETP **9**, 374 (1959).

² J. Blomqvist and S. Wahlhorn, Ark. Fysik **16**, 545 (1960).

³ A. Bohr and B. Mottelson, Mat.-Fys. Medd. Dan. Vid. Selsk. **27**, No. 16, (1953).

⁴ B. J. Raz, Phys. Rev. **114**, 1116 (1959).

⁵ B. J. Raz, Phys. Rev. **120**, 169 (1960).

⁶ Guman, Sliv, and Sogomonova, JETP **40**, 341 (1961), Soviet Phys. JETP **13**, 232 (1961).

Translated by J. G. Adashko
140

EFFECT OF ZERO-POINT SHAPE VIBRATIONS OF HEAVY NUCLEI ON ALPHA-DECAY PROBABILITIES

V. G. NOSOV

Submitted to JETP editor March 21, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 806-809 (September, 1961)

The alpha decay theory previously developed for nonspherical nuclei is used to determine a new shape parameter, the amplitude of the zero-point vibration of the quadrupole deformation. It is shown that different experiments actually measure different deformation values. In particular, the difference between the quadrupole deformation determined from alpha decay and the equilibrium quadrupole deformation can be expressed in terms of the mean square fluctuation of the ground state deformation of the nucleus. This mean square fluctuation is determined from the experimental data for 21 even-even nuclei. It is shown that the amplitude of zero-point quadrupole deformation vibrations has a sharp maximum at the boundary between spherical and nonspherical nuclei.

MEASUREMENTS of various physical nuclear properties that depend on the quadrupole deformation α_2 are frequently dealt with at present. There are zero-point vibrations of this deformation in the ground state of the internal nuclear motion. In fact, according to quantum mechanics^[1] the deformation distribution function has the form

$$f(\alpha_2) = (2\pi\bar{x}^2)^{-1/2} \exp\{-x^2/2\bar{x}^2\}, \quad (1)$$

where $x = \alpha_2 - \bar{\alpha}_2$ with $\bar{\alpha}_2$ the equilibrium deformation. There is therefore a question as to which values of the quadrupole deformation are involved in the results of such measurements.

In many experiments (Coulomb excitation, lifetimes of excited states, etc.) the measured quantity is the internal quadrupole moment of the nonspherical nucleus

$$Q_0 = \frac{6}{5} ZR_0^2 \alpha_2. \quad (2)$$

Multiplying by (1) and integrating over α_2 gives again Eq. (2) but with α_2 replaced by $\bar{\alpha}_2$. Thus, it is the equilibrium deformation $\bar{\alpha}_2$ that is involved in the measurements of the electric quadrupole moment of a nonspherical nucleus.

The situation is different in the theory of alpha decay.^[2] According to Eq. (2.18) of^[2], the probability amplitude for emission of an alpha particle with angular momentum l depends on the quadrupole deformation of the daughter nucleus through a factor $X_l(\beta)$. The true probability amplitude is proportional to the integral

$$\int X_l(\beta) f(\alpha_2) d\alpha_2.$$

With the aid of Eq. (2.23) of^[2], it is not difficult to

see that $X_l(\beta) = e^\beta F_l(\beta)$, where $F_l(\beta)$ is a slowly varying function (in the region of deformation values of interest it is sufficient to retain the first few terms in the asymptotic expansion of F_l in inverse powers of β).

Thus, the dependence of the probability amplitude on the deformation is essentially determined by the factor $\exp(b\alpha_2)$, where $b \equiv \beta/\alpha_2$ is given by Eq. (2.23) of^[2]. We transform the integrand so as to reveal the position of its maximum:

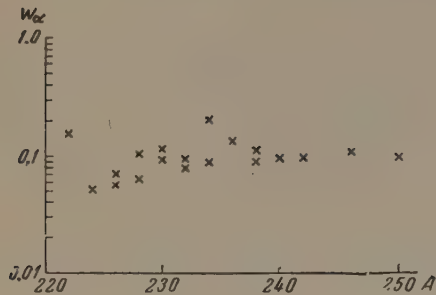
$$e^{b\alpha_2} f(\alpha_2) = \exp\{-b^2\bar{x}^2/2 + b\alpha_2^{(\alpha)}\} (2\pi\bar{x}^2)^{-1/2} \exp\{-x_\alpha^2/2\bar{x}^2\}; \quad (3)$$

$$x_\alpha = \alpha_2 - \alpha_2^{(\alpha)}, \quad \alpha_2^{(\alpha)} = \bar{\alpha}_2 + b\bar{x}^2. \quad (4)$$

We see that the exponential dependence of the alpha decay probability amplitude on the deformation shifts the maximum of the integrand to the point $\alpha_2 = \alpha_2^{(\alpha)}$. Consequently, it is not the equilibrium deformation $\bar{\alpha}_2$ that enters everywhere in the formulas of the theory of alpha decay,^[2] but another deformation $\alpha_2^{(\alpha)}$ defined by Eq. (4).

Since the lifetimes of the rotational 2^+ states in many heavy even-even nuclei have been measured by Bell et al,^[3] it is possible to use Eq. (4) to compute the corresponding zero-point vibration amplitudes $(\bar{x}^2)^{1/2}$. The results are listed in the table. The deformation $\alpha_2^{(\alpha)}$ was determined from the alpha decay fine structure and Eq. (2.22)^[2], by the procedure described in^[4]. The values of $\bar{\alpha}_2$ are taken from^[3]. The first lines of the table show nuclei near the boundary between the regions of spherical and nonspherical nuclei. The nucleus Ra^{222} is apparently of an intermediate type, and the theory of nonspherical nuclei is not fully ap-

Nu- cleus	$\alpha_2^{(\alpha)}$	$\bar{\alpha}_2$	$(\bar{x}^2)^{1/2}$	$(\bar{x}^2)^{1/2} / \bar{\alpha}_2$	Nu- cleus	$\alpha_2^{(\alpha)}$	$\bar{\alpha}_2$	$(\bar{x}^2)^{1/2}$	$(\bar{x}^2)^{1/2} / \bar{\alpha}_2$
Ra ²²²	0.18	0.12	0.07	0.6	Th ²³⁴	0.17	0.15	0.04	0.3
Ra ²²⁴	0.24	0.11	0.10	0.9	U ²³⁰	0.15	0.15	~0	~0
Ra ²²⁶	0.22	0.12	0.08	0.7	U ²³²	0.22	0.16	0.07	0.4
Ra ²²⁸	0.21	0.13	0.08	0.6	U ²³⁴	0.20	0.16	0.06	0.4
Th ²²⁶	0.23	0.14	0.08	0.6	U ²³⁶	0.17	0.17	~0	~0
Th ²²⁸	0.21	0.14	0.07	0.5	U ²³⁸	0.16	0.17	~0	~0
Th ²³⁰	0.19	0.15	0.06	0.4	Pu ²³⁸	0.19	0.17	0.04	0.2
Th ²³²	0.19	0.15	0.05	0.3	Pu ²⁴⁰	0.17	0.17	~0	~0



plicable to it.* As is evident from the last column of the table, the quantum fluctuations of the quadrupole deformation are quite significant near the boundary of the region of stability of nonspherical shape.

An independent estimate of the zero-point vibration amplitude can be obtained from the probabilities w_α for formation of alpha particles within a nucleus. These probabilities are determined from the observed half-lives.^[4,5] It is easy to see that the effect of the zero-point deformation vibrations is to renormalize these internal formation probabilities. Integrating both sides of Eq. (3) over α_2 and squaring, we obtain

$$w_\alpha = \tilde{w}_\alpha \exp(-b^2 \bar{x}^2), \quad (\bar{x}^2)^{1/2} = b^{-1} \ln^{1/2}(\tilde{w}_\alpha / w_\alpha), \quad (5)$$

where $\tilde{w}_\alpha$ is the true internal formation probability and w_α is the experimentally observed renormalized internal formation probability, which is determined from the experimental data by the procedure described in references 4 and 5. The values of w_α for twenty even-even nuclei are plotted against mass number in the figure. The general trend of the internal formation probabilities reflects some increase of the renormalization logarithm $\ln(w_\alpha / \tilde{w}_\alpha)$ with decreasing mean square quadrupole deformation fluctuation. Unfortunately, the internal formation probability fluctuates from nucleus to nucleus, so that we had to average $\ln w_\alpha$ over several nuclei in order to use Eq. (5). In order to estimate the zero-point vibration amplitude $(x^2)^{1/2}$ we averaged $\ln w_\alpha$ over the five nuclei Ra^{224,226,228} and Th^{226,228} (for

the reason mentioned above, the extreme left point of the figure, corresponding to Ra²²², was omitted from consideration) and, assuming that there is no renormalization in the other fourteen, we obtained a corresponding average of $\ln \tilde{w}_\alpha$. Substitution in Eq. (5) gave $(x^2)^{1/2} = 0.05$. The fact that this estimate is somewhat lower than the values in the table is probably due to the neglect of the zero-point vibrations in the region $A > 228$.

In conclusion, we note that the zero-point deformation vibration amplitude also rises sharply if the nonspherical shape stability boundary is approached from the region of spherical nuclei. This is easily seen from the experimental data on electric quadrupole $2^+ \rightarrow 0^+$ transition probabilities for isotopes of radon^[3] and polonium.^[6] The quadrupole vibrations of spherical nuclei are fivefold degenerate.^[6] If, of the five degrees of freedom, we consider just the one corresponding to the oscillation which does not destroy the axial symmetry of the nuclear shape with respect to some direction in space, then the amplitude of the zero-point vibrations of the corresponding deformation is determined from the relation

$$w = \frac{3}{125} \frac{e^2 \omega^5}{\hbar c^5} Z^2 R_0^4 \bar{x}^2, \quad (6)$$

where w is the probability per unit time for emission of the nuclear gamma from the 2^+ first excited state, $\hbar\omega$ is the energy of the gamma, and $\bar{x}^2 \equiv \alpha_2^2$ is the mean square quadrupole deformation fluctuation in the ground state of the nucleus and is actually the measured quantity in this case.

The mean square fluctuations $(\alpha_2^2)^{1/2}$ calculated with an assumed value $R_0 = 1.2 A^{1/3}$ Fermi units, are:

Nucleus	Po ²¹²	Po ²¹⁴	Em ²¹⁰	Em ²¹¹	Em ²²²
$(\alpha_2^2)^{1/2}$:	0.015	0.021	>0.025	0.035	0.039

It is evident that large quantum fluctuations of the quadrupole deformation are a characteristic feature of the nuclear "phase transition" corresponding to a change in the symmetry of the equilibrium shape.

*In this connection, see below, and also the seventh footnote of [4].

- ¹ L. I. Schiff, *Quantum Mechanics* (2nd ed.), McGraw Hill, New York 1955.
- ² V. G. Nosov, JETP **39**, 141 (1960), Soviet Phys. JETP **12**, 102 (1961).
- ³ Bell, Bjørnholm, and Severiens, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **32**, no. 12 (1960).
- ⁴ V. G. Nosov, JETP **39**, 1660 (1960), Soviet Phys. JETP **12**, 1159 (1961).
- ⁵ V. G. Nosov, JETP **36**, 1580 (1959), Soviet Phys. JETP **9**, 1122 (1959).
- ⁶ Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. **28**, 432 (1956).

Translated by M. Bolsterli
141

SYMMETRY PROPERTIES OF STRONG INTERACTIONS

V. M. SHEKHTER

Leningrad Physico-Technical Institute

Submitted to JETP editor March 24, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 810-820 (September, 1961)

A method is proposed which makes it possible to find all symmetry properties of the strong interactions which are possible at least in principle, provided the most general form of the Lagrangian is given (in the sense of the values of the bare coupling constant and particle masses). The method is applied to the Lagrangian in the Yukawa form and to a compound model with a four-fermion interaction.

1. INTRODUCTION

IN recent years many papers have been devoted to the study of the possible symmetry properties of the strong interactions. Many types of symmetry have been proposed, but all of them, with the exceptions of the conservation of the number of baryons, and of strangeness and isotopic invariance, are to some extent in contradiction with experiment. But both the question as to whether some types of symmetry can hold approximately and that of the nature of the approximation involved in such a case are still unclear. This is due to the fact that as a rule it is only by a difficult quantitative analysis that one can distinguish a violation of a symmetry property that is due to its approximate nature from the complete absence of the property. Nevertheless, the question of the symmetries of strong interactions remains as interesting as before, and the search for such properties continues.

Theoretical work devoted to the invariance properties of the strong interactions is most often done in the following way. One writes down a Lagrangian "in the most general form." Its free-particle part contains the bare masses of the particles, and the interaction contains the coupling constants. Both sets of quantities are essentially unknown parameters of the theory. The next step is to equate some of these parameters to each other, and some to zero. The new Lagrangian has additional symmetry properties as compared with the original Lagrangian.

The change from the "general" to the "special" Lagrangian is more natural in some cases than in others, and is usually due to the inventiveness of the author. In the present paper we propose a rather simple and in some sense standard method

for this change, which guarantees the obtaining of all "special" Lagrangians which are possible in principle and of the corresponding symmetry properties, as soon as the Lagrangian of most general form has been written down. This method has been used earlier by the writer,^[1] and has also been used recently by Behrends and Sirlin.^[2]

The method is applied to the Lagrangian written in the Yukawa form. If we assume that the parities of Λ and Σ , and also of N and Ξ , are the same, and that the parities of K and Λ are different (other possibilities are discussed in Sec. 4), then in the most general case the strong-interaction Lagrangian which conserves baryon number, strangeness, and electric charge, and also satisfies the requirement of isotopic invariance, has the well known form

$$L = \{g_1(\bar{N}i\gamma_5\tau N) + g_2(\bar{\Lambda}i\gamma_5\Sigma + \bar{\Sigma}i\gamma_5\Lambda) - ig_3[\bar{\Sigma}i\gamma_5\Sigma] + g_4(\bar{\Xi}i\gamma_5\tau\Xi)\}\pi + \{g_5(\bar{N}i\gamma_5\Lambda)K + g_6(\bar{N}i\gamma_5\tau\Sigma)K + g_7K(\bar{\Lambda}i\gamma_5i\tau_2\Xi) + g_8K(\bar{\Sigma}i\gamma_5i\tau_2\tau\Xi) + \text{Herm. adj.}\}, \quad (1)$$

where N , Λ , and so on denote the operators for the corresponding particles in the Heisenberg representation, the bold-face letters are isotopic vectors, τ_i are the isotopic Pauli matrices, $\pi^\pm = (\pi_1 \mp i\pi_2)/2^{1/2}$, $\Sigma^\pm = (\Sigma_1 \mp i\Sigma_2)/2^{1/2}$, and g_i are the real, and in general unequal, bare strong-interaction constants.

If there exists in nature a ρ meson which is pseudoscalar in ordinary space and scalar in isotopic space, then we must add to L

$$L' = \{g_9(\bar{N}i\gamma_5N) + g_{10}(\bar{\Lambda}i\gamma_5\Lambda) + g_{11}(\bar{\Sigma}i\gamma_5\Sigma) + g_{12}(\bar{\Xi}i\gamma_5\Xi)\}\rho. \quad (2)$$

The ρ meson may decay very rapidly, and therefore be practically unobservable, but the addition of the expression (2) to the Lagrangian (1) in-

creases the number of parameters g_j which can be varied, and consequently increases the number of classes of symmetry possible in principle.

As has already been remarked, for certain relations between the g_j , and also between the bare masses, which appear in the free-particle Lagrangian, the theory is invariant under corresponding transformations of the operators for the particles. The problem is to find all possible relations of this kind and the corresponding classes of symmetry of the theory with the Lagrangian (1)–(2).*

The method proposed for the solution of this problem is essentially very simple. As is well known, to every symmetry of the theory there must correspond a certain current j_λ , bilinear in the operators that appear in Eqs. (1) and (2), and having a vanishing divergence ($\partial j_\lambda / \partial x_\lambda = 0$). Therefore the finding of all possible symmetry classes is equivalent to the problem of finding all the currents whose divergences vanish under various assumptions about the values of the constants g_j and masses m_i . Because of the isotopic invariance of the theory these currents must be either scalars, or vectors, or spinors in the isospace. When written in the most general form, each current of this type contains a certain number of arbitrary constants. By calculating the divergence of this current by means of the equations of motion, which are direct consequences of Eqs. (1) and (2), and equating it to zero, one can get a system of homogeneous algebraic equations for these constants, and thus for the g_j and m_i . By finding all the non-trivial solutions of such a system one can solve the problem of constructing currents with vanishing divergence, and consequently also the problem of determining the possible symmetry classes of the theory with the Lagrangian (1)–(2).

Among the solutions obtained in this way will of course be the “obvious” ones, which are valid for arbitrary g_j and m_i . They correspond to conservation of baryon number, strangeness, and charge and isotopic invariance, since these properties of the theory are already contained in the expressions (1) and (2). Strictly speaking all of the “nonobvious” solutions are in contradiction with experiment, because if they hold then, as will be shown below, either certain observed processes must be forbidden, or the masses of different particles must coincide, which again disagrees with experi-

ment. Therefore “nonobvious” symmetry properties can hold only approximately. As has already been noted, the question of the degree of validity of such an approximation is a difficult one, which must be given separate consideration in each concrete case and is not in the scope of this paper. We can only remark at once that the symmetry classes considered earlier by Gell-Mann,^[3] Pais,^[4] and other authors are among the “nonobvious” solutions.

As has already been noted, the problem of finding the “nonobvious” solutions has been treated in a paper by Behrends and Sirlin.^[2] These authors do not introduce the ρ meson into the Lagrangian and confine themselves to solutions that hold either when all the g_i ($i = 1, 2, \dots, 8$) are different from zero or when $g_5 = g_6 = g_7 = g_8 = 0$, $g_1, g_2, g_3, g_4 \neq 0$. For this reason they do not get the solutions given in Eqs. (19), (20), (30), (31), and (32) and in the Appendix. In the remaining cases our results agree.

In the next section we shall find in explicit form the currents that conserve strangeness and have zero divergence. For each such current there are indicated, at least partially, the symmetry properties following from the equation $\partial j_\lambda / \partial x_\lambda = 0$ that are in contradiction with experiment (without counting the ascribing of equal masses to different particles). An analogous study of the properties of currents with parity changes is made in Sec. 3. The results obtained are discussed briefly in Sec. 4. The problem of the connection of the results of Sec. 2 and 3 with the properties of conservation of the vector currents involved in the weak interactions is not considered, since the writer has already devoted a paper to this.^[1] The symmetry properties of the strong interactions in the Sakata-Okun’ model^[5–7] are investigated in Sec. 5.

2. CURRENTS WITH CONSERVATION OF STRANGENESS

If we begin with isotopic scalar currents, in the most general form they can be written as follows (a_i are arbitrary constants):

$$j_\lambda^S = a_1 (\bar{N} \gamma_\lambda N) + a_2 (\bar{\Lambda} \gamma_\lambda \Lambda) + a_3 (\bar{\Sigma} \gamma_\lambda \Sigma) + a_4 (\bar{\Xi} \gamma_\lambda \Xi) - a_5 (\bar{K} \partial_\lambda K), \quad (3)$$

where the symbol ∂_λ has the meaning

$$(\Phi_2 \partial_\lambda \Phi_1) \equiv \Phi_2 \frac{\partial \Phi_1}{\partial x_\lambda} - \frac{\partial \Phi_2}{\partial x_\lambda} \Phi_1. \quad (4)$$

A direct calculation of the divergence of the current (3) with the use of Eqs. (1) and (2) leads to the expression

*It must be stated at the start that we are concerned with invariance properties of the strong interactions under continuous transformations of the type of rotations in the isospace; discrete transformations of the type of reflections are not considered.

$$\begin{aligned}
\partial j_\lambda^S / \partial x_\lambda = & [(\bar{\Lambda} i \gamma_5 \Sigma) \pi - \pi (\bar{\Sigma} i \gamma_5 \Lambda)] g_2 (a_2 - a_3) \\
& + [(\bar{N} i \gamma_5 \Lambda) K - \bar{K} (\bar{\Lambda} i \gamma_5 N)] g_5 (a_1 - a_2 - a_5) \\
& + [(\bar{N} i \gamma_5 \tau \Sigma) K - \bar{K} (\bar{\Sigma} i \gamma_5 N)] g_6 (a_1 - a_3 - a_5) \\
& + [K (\bar{\Lambda} i \tau_2 i \gamma_5 \Xi) + (\bar{\Xi} i \tau_2 i \gamma_5 \Lambda) \bar{K}] g_7 (a_2 - a_4 - a_5) \\
& + [K (\bar{\Sigma} i \tau_2 i \gamma_5 \Xi) + (\bar{\Xi} i \tau_2 i \gamma_5 \Sigma) \bar{K}] g_8 (a_3 - a_4 - a_5). \quad (5)
\end{aligned}$$

If we require that $\partial j_\lambda^S / \partial x_\lambda = 0$, then in Eq. (5) we must equate to zero the coefficients of all derivatives of Heisenberg operators, since these derivatives are independent. This gives the system of algebraic equations

$$\begin{aligned}
g_2 (a_2 - a_3) = 0, \quad g_5 (a_1 - a_2 - a_5) = 0, \\
g_6 (a_1 - a_3 - a_5) = 0, \\
g_7 (a_2 - a_4 - a_5) = 0, \quad g_8 (a_3 - a_4 - a_5) = 0. \quad (6)
\end{aligned}$$

The system (6) is satisfied for arbitrary g_j if

$$a_2 = a_3 = a_1 - a_5 = a_4 - a_5. \quad (7)$$

The relation (7) gives three equations for five unknowns, and consequently has two linearly independent solutions, which we can take to be

$$\begin{aligned}
1) \quad a_1 = 1, \quad a_5 = 0, \quad a_2 = a_3 = a_4 = 1, \\
2) \quad a_1 = 0, \quad a_5 = 1; \quad a_2 = a_3 = a_4 / 2 = -1. \quad (8)
\end{aligned}$$

Thus for arbitrary values of g_j we see that currents with vanishing divergence are

$$\begin{aligned}
j_\lambda^S = (\bar{N} \gamma_\lambda N) + (\bar{\Lambda} \gamma_\lambda \Lambda) + (\bar{\Sigma} \gamma_\lambda \Sigma) + (\bar{\Xi} \gamma_\lambda \Xi), \\
j_\lambda^S = -(\bar{\Lambda} \gamma_\lambda \Lambda) - (\bar{\Sigma} \gamma_\lambda \Sigma) - 2(\bar{\Xi} \gamma_\lambda \Xi) - (\bar{K} \partial_\lambda K), \quad (9)
\end{aligned}$$

which corresponds to the conservation of baryon number and strangeness, which was obvious from the beginning.

If we return to Eq. (6) and exclude the obvious solutions (8)—to do so it is enough to set $a_1 = a_5 = 0$ —then there remains

$$g_2 (a_2 - a_3) = g_5 a_2 = g_6 a_3 = g_7 (a_2 - a_4) = g_8 (a_3 - a_4) = 0. \quad (10)$$

Any solution of Eq. (10) is in contradiction with experiment. In fact, if in Eq. (10) $g_5 = g_6 = 0$ or $g_7 = g_8 = 0$, then either the KN or the $K\Xi$ interactions are absent. This means that either the number of particles N or the number of Ξ is conserved, and this does not agree with experiment. The other possibilities in Eq. (10) are

$$\begin{aligned}
1) \quad a_2 = 0, \quad a_3 \neq 0, \quad g_2 = g_6 = 0, \quad a_4 = g_8 = 0, \\
2) \quad a_2 = 0, \quad a_3 \neq 0, \quad g_2 = g_6 = 0, \quad g_7 = a_3 - a_4 = 0, \\
3) \quad a_3 = 0, \quad a_2 \neq 0, \quad g_2 = g_5 = 0, \quad a_4 = g_7 = 0, \\
4) \quad a_3 = 0, \quad a_2 \neq 0, \quad g_2 = g_5 = 0, \quad g_8 = a_2 - a_4 = 0. \quad (11)
\end{aligned}$$

These four cases correspond to constancy of N_Σ , $N_\Sigma + N_\Xi$, N_Λ ; and $N_\Lambda + N_\Xi$ (N_R is the number of particles R). This means that observed reactions of the types $K^- + p \rightarrow \Sigma^\pm + \pi^\mp$ or $K^- + p \rightarrow \Lambda + \pi^0$ would be forbidden. Consequently,

strictly speaking, there can be no isoscalar currents except (9) which have vanishing divergence.

If we now go on to isovector currents, in the most general case we have

$$\begin{aligned}
j_\lambda^V = & b_1 (\bar{N} \gamma_\lambda \tau N) + b_2 (\bar{\Lambda} \gamma_\lambda \Sigma + \bar{\Sigma} \gamma_\lambda \Lambda) - ib_3 [\bar{\Sigma} \gamma_\lambda \cdot \Sigma] \\
& + b_4 (\bar{\Xi} \gamma_\lambda \tau \Xi) - ib_5 \left[\pi, \frac{\partial}{\partial x_\lambda} \pi \right] - b_6 (\bar{K} \tau \partial_\lambda K) \\
& - b_7 (\rho \partial_\lambda \pi) + b_8 (\bar{\Lambda} \gamma_\lambda \Sigma - \bar{\Sigma} \gamma_\lambda \Lambda). \quad (12)
\end{aligned}$$

As in the case of Eqs. (5) and (6), the equation $\partial j_\lambda^V / \partial x_\lambda = 0$ leads to the system of equations

$$\begin{aligned}
b_7 g_1 = b_7 g_2 = b_7 g_3 = b_7 g_4 = b_7 g_9 = b_7 g_{10} = b_7 g_{11} = b_7 g_{12} = 0, \\
b_8 g_2 = b_8 g_5 = b_8 g_6 = b_8 g_7 = b_8 g_8 = b_8 (g_{10} - g_{11}) \\
= b_8 (m_\Sigma - m_\Lambda) = 0, \\
(2b_1 + b_5) g_1 = 0, \quad (b_1 - b_5) g_5 - b_2 g_6 = 0, \\
(2b_4 + b_5) g_4 = 0, \quad (b_1 - b_5) g_6 - b_2 g_5 = 0, \\
(b_3 + b_5) g_2 + b_2 g_3 = 0, \quad g_6 (b_1 + b_5 - b_3) = 0, \\
(b_3 + b_5) g_3 + b_2 g_2 = 0, \quad (b_4 - b_5) g_7 - b_2 g_8 = 0, \\
b_2 (m_\Sigma - m_\Lambda) = 0, \quad (b_4 - b_5) g_8 - b_2 g_7 = 0, \\
b_2 (g_{10} - g_{11}) = 0, \quad g_8 (b_4 + b_5 - b_3) = 0. \quad (13)
\end{aligned}$$

The system (13) has the obvious solution

$$\begin{aligned}
b_1 = b_3 / 2 = b_4 = -b_5 / 2 = b_6, \\
b_2 = b_7 = b_8 = 0, \quad (14)
\end{aligned}$$

since the vanishing of the divergence of the current

$$\begin{aligned}
j_\lambda^V = & (\bar{N} \gamma_\lambda \tau N) - 2i [\bar{\Sigma} \gamma_\lambda \cdot \Sigma] + (\bar{\Xi} \gamma_\lambda \tau \Xi) \\
& + 2i [\pi \partial_\lambda \pi / \partial x_\lambda] - (\bar{K} \tau \partial_\lambda K) \quad (15)
\end{aligned}$$

(instead of $2[\pi \partial_\lambda \pi / \partial x_\lambda]$ we can of course write $[\pi, \partial_\lambda \pi]$) is equivalent to electric charge conservation and isotopic invariance. Components of the current (15) occur in the Lagrangian of the electromagnetic interaction and in the scheme of weak interactions proposed by Feynman and Gell-Mann.^[8]

In order to eliminate from Eq. (13) the obvious solution (14), we can set $b_5 = 0$ in Eq. (13). In the solution of the system that results from this we shall not consider cases in which even any one of the equations

$$\begin{aligned}
g_5 = g_6 = 0, \quad g_2 = g_5 = g_7 = 0, \quad g_2 = g_6 = g_8 = 0, \\
g_7 = g_8 = 0, \quad g_1 = g_2 = g_3 = g_4 = 0, \quad (16)
\end{aligned}$$

holds; these equations mean the unobserved conservation of the numbers of particles N, Λ , Σ , Ξ , or π , respectively. The cases (16) can be considered separately; the results of such a treatment are given in the Appendix (the fact that the cross section for Ξ production is small experimentally^[9] might indeed mean that g_7 and g_8 are small; the vanishing of g_5 , g_6 , g_7 , g_8 corresponds approximately to the "very strong" coupling of Gell-Mann^[3]). After the possibilities (16) are ruled out

the system (13) with $b_5 = 0$ has only three solutions:

$$1) \quad g_2 = \varepsilon g_3, \quad g_5 = \varepsilon g_6, \quad g_7 = \varepsilon g_8, \quad \varepsilon = \pm 1; \quad g_{10} = g_{11}, \\ m_\Lambda = m_\Sigma, \quad b_7 = b_8 = b_1 = b_4 = 0, \quad b_3 = b_6 = -\varepsilon b_2, \quad (17)$$

$$j_\lambda^{V_1} = -\varepsilon (\bar{\Lambda} \gamma_\lambda \Sigma + \bar{\Sigma} \gamma_\lambda \Lambda) - i [\bar{\Sigma} \gamma_\lambda \cdot \Sigma] - (\bar{K} \tau \partial_\lambda K).$$

$$2) \quad g_1 = g_4 = 0, \quad g_2 = -\varepsilon g_3, \quad g_5 = \varepsilon g_6, \quad g_7 = \varepsilon g_8, \\ \varepsilon = \pm 1, \quad g_{10} = g_{11}, \quad m_\Lambda = m_\Sigma;$$

$$b_7 = b_8 = b_6 = 0, \quad b_1 = \varepsilon b_2 = b_3 = b_4, \quad (18)$$

i.e., with $b_1 = 1$

$$j_\lambda^{V_2} = (\bar{N} \gamma_\lambda \tau N) + \varepsilon (\bar{\Lambda} \gamma_\lambda \Sigma + \bar{\Sigma} \gamma_\lambda \Lambda) - i [\bar{\Sigma} \gamma_\lambda \cdot \Sigma] + (\bar{\Xi} \gamma_\lambda \tau \Xi).$$

$$3) \quad g_1 = g_4 = g_6 = g_8 = 0, \quad b_7 = b_8 = b_2 = b_3 = 0, \\ b_1 = b_4 = b_6, \quad (19)$$

i.e., with $b_1 = 1$

$$j_\lambda^{V_3} = (\bar{N} \gamma_\lambda \tau N) + (\bar{\Xi} \gamma_\lambda \tau \Xi) - (\bar{K} \tau \partial_\lambda K).$$

The appearance of sign factors ε in Eqs. (17) and (18) and in subsequent formulas is due to the possibility of arbitrary changes of sign of the operators for the particles and of the strong-interaction constants. The question of the physical meaning of the relative signs of these constants has been discussed by Kobzarev and Okun'.^[10]

The current $j_\lambda^{V_1}$ and the conditions on g_j in Eq. (17) are equivalent to the symmetry of Pais,^[4] and the author has discussed them earlier^[11] in more detail [in^[11] the signs of all the baryon currents must be changed, and also the sign of ε' in Eq. (17)]. Once again, strictly speaking, the conditions for the constants g_j in Eqs. (18) and (19) are in contradiction with experiment. In fact, the vanishing of the divergences of the currents $j_\lambda^{V_2}$ and $j_\lambda^{V_3}$ means the conservation of the respective operators

$$\hat{N}_p - \hat{N}_n - \hat{N}_{\Sigma^+} + \varepsilon (\hat{N}_{\Sigma^0} + \hat{N}_{\Sigma^-}) - \hat{N}_{\Sigma^-} + \hat{N}_{\Xi^0} - \hat{N}_{\Xi^-}, \\ \hat{N}_p - \hat{N}_n + \hat{N}_{\Xi^0} - \hat{N}_{\Xi^-} + \hat{N}_{K^+} - \hat{N}_{K^0}, \quad (20)$$

where the particle-number operators $\hat{N}_\psi$ and $\hat{N}_\Phi$ for the spinor ψ and the boson Φ are defined by $\hat{N}_\psi = \int \psi^\dagger(x) \psi(x) d^3x$, $\hat{N}_\Phi = -\int \Phi^\dagger(x) \partial_4 \Phi(x) d^3x$, (21)

and $Y^0 = (\Lambda^0 - \Sigma^0)/2^{1/2}$, $Z^0 = (\Lambda^0 + \Sigma^0)/2^{1/2}$. The conservation of either of the operators (20) forbids, for example, π -meson charge transfer ($\pi^- + p \rightarrow \pi^0 + n$) and a number of other observed reactions.

3. CURRENTS WITH CHANGE OF STRANGENESS

An isoscalar current with change of strangeness by two units can consist of only two terms:

$$j_\lambda^B = c_1 (\bar{N} \gamma_\lambda \Xi) + \frac{1}{2} c_2 (\bar{K} i \tau_2 \partial_\lambda \bar{K}) \\ = c_1 (\bar{N} \gamma_\lambda \Xi) + c_2 (\bar{K} i \tau_2 \partial \bar{K} / \partial x_\lambda). \quad (22)$$

The divergence condition $\partial j_\lambda^B / \partial x_\lambda = 0$ holds for

$$c_1 (m_N - m_\Xi) = c_2 m_K^2 = 0, \quad c_1 (g_1 - g_4) = c_1 (g_9 - g_{12}) = 0, \\ c_1 g_5 - c_2 g_7 = 0, \quad c_1 g_6 - c_2 g_8 = 0, \quad c_1 g_7 - c_2 g_5 = 0, \\ c_1 g_8 - c_2 g_6 = 0. \quad (23)$$

The system (23) has two solutions:

$$a) \quad g_5 = \varepsilon g_7, \quad g_6 = \varepsilon g_8, \quad \varepsilon = \pm 1; \quad g_1 = g_4, \quad g_9 = g_{12},$$

$$m_N = m_\Xi, \quad m_K = 0; \quad c_1 = \varepsilon c_2,$$

$$b) \quad g_5 = g_6 = g_7 = g_8 = 0, \quad g_1 = g_4, \quad g_9 = g_{12}, \quad m_N = m_\Xi; \\ c_1 \neq 0, \quad (24)$$

the first of which has been considered earlier by Feinberg and Behrends.^[11] The second solution requires the absence of K-meson interactions.

The only isovector current with change of strangeness by 2, $(\bar{N} \gamma_\lambda \tau \Xi) [(\bar{K} i \tau_2 \tau \partial_\lambda \bar{K}) \equiv 0]$, has zero divergence only under the senseless condition that N and Ξ have no strong interactions.

Currents with strangeness change by unity can have only half-integral isotopic spins ($1/2$ or $3/2$), since only expressions of the form $(\bar{N} \gamma_\lambda \Lambda)$ or $(\bar{N} \gamma_\lambda \Sigma)$ can be constructed from the quantities of the theory.

Let us first consider the current with isospin $1/2$. In the most general case it is of the form

$$j_\lambda^\psi = d_1 (\bar{N} \gamma_\lambda \Lambda) + d_2 (\bar{N} \gamma_\lambda \tau \Sigma) + d_3 (\bar{\Lambda} \gamma_\lambda i \tau_2 \Xi) \\ + d_4 (\bar{\Sigma} \gamma_\lambda i \tau_2 \tau \Xi) + d_5 (\bar{K} \tau \partial_\lambda \pi) + d_6 (\bar{K} \partial_\lambda \rho), \quad (25)$$

or, in terms of components

$$(j_\lambda^\psi)_{1/2} = d_1 (\bar{p} \gamma_\lambda \Lambda) + d_2 [(\bar{p} \gamma_\lambda \Sigma^0) + \sqrt{2} (\bar{n} \gamma_\lambda \Sigma^-)] + d_3 (\bar{\Lambda} \gamma_\lambda \Xi^-) \\ + d_4 [-(\bar{\Sigma}^0 \gamma_\lambda \Xi^-) + \sqrt{2} (\bar{\Sigma}^+ \gamma_\lambda \Xi^0)] + d_5 [(\bar{K}^+ \partial_\lambda \pi^0) \\ + \sqrt{2} (\bar{K}^0 \partial_\lambda \pi^-)] + d_6 (\bar{K}^+ \partial_\lambda \rho), \quad (26)$$

$$(j_\lambda^\psi)_{-1/2} = d_1 (\bar{n} \gamma_\lambda \Lambda) + d_2 [-(\bar{n} \gamma_\lambda \Sigma^0) + \sqrt{2} (\bar{p} \gamma_\lambda \Sigma^+)] \\ - d_3 (\bar{\Lambda} \gamma_\lambda \Xi^0) + d_4 [-(\bar{\Sigma}^0 \gamma_\lambda \Xi^0) - \sqrt{2} (\bar{\Sigma}^- \gamma_\lambda \Xi^-)] \\ + d_5 [-(\bar{K}^0 \partial_\lambda \pi^0) + \sqrt{2} (\bar{K}^+ \partial_\lambda \pi^+)] + d_6 (\bar{K}^0 \partial_\lambda \rho). \quad (27)$$

Just as in the preceding cases, the requirement $\partial j_\lambda^\psi / \partial x_\lambda = 0$ leads to a system of equations

$$\begin{aligned} -d_1 g_1 + d_2 g_2 + d_5 g_5 &= 0, & -d_3 g_4 + d_4 g_2 + d_5 g_7 &= 0, \\ d_1 g_2 - d_2 g_1 + d_5 g_6 &= 0, & d_3 g_2 - d_4 g_4 + d_5 g_8 &= 0, \\ d_1 g_5 - d_2 g_6 - 2d_5 g_1 &= 0, & d_3 g_7 - d_4 g_8 - 2d_5 g_4 &= 0, \\ d_2 (g_1 - g_3) + d_5 g_6 &= 0, & d_4 (g_4 - g_3) + d_5 g_9 &= 0, \\ -d_2 g_6 - d_4 g_8 + d_5 g_3 &= 0, & d_1 (g_{10} - g_9) + d_5 g_5 &= 0, \\ d_2 g_5 + d_3 g_8 + d_5 g_2 &= 0, & d_2 (g_{11} - g_9) + d_5 g_6 &= 0, \\ d_1 g_6 + d_4 g_7 + d_5 g_2 &= 0, & d_3 (g_{12} - g_{10}) + d_5 g_7 &= 0, \\ d_1 g_7 + d_2 g_8 + 2d_4 g_6 &= 0, & d_4 (g_{12} - g_{11}) + d_5 g_8 &= 0, \\ 2d_2 g_8 + d_3 g_5 + d_4 g_6 &= 0, & 2d_2 g_6 + d_5 g_1 - d_5 g_9 &= 0, \\ -d_1 g_5 + d_3 g_7 - d_5 g_{10} &= 0, & -d_2 g_6 + d_4 g_8 - d_5 g_{11} &= 0, \\ d_5 (m_\pi - m_K) &= d_5 (m_\rho - m_K) = 0, & 2d_4 g_8 + d_5 g_4 - d_5 g_{12} &= 0, \\ d_1 (m_N - m_\Lambda) &= d_2 (m_N - m_\Sigma) = d_3 (m_\Lambda - m_\Xi) \\ &= d_4 (m_\Sigma - m_\Xi) = 0. \end{aligned} \quad (28)$$

The solution of the system (28) is rather cumbersome, although elementary. Therefore we give here only the final result. If we confine ourselves to the condition that Eqs. (16) do not hold (otherwise there are too many solutions), the divergence of the current j_λ^ψ is zero in only four cases:

$$1) \quad g_1 = -\varepsilon g_2 = -g_3 = g_4 = \varepsilon'' g_5 = \varepsilon \varepsilon'' g_6 = \varepsilon' \varepsilon'' g_7 \\ = \varepsilon \varepsilon' \varepsilon'' g_8, \quad \varepsilon, \varepsilon', \varepsilon'' = \pm 1, \\ g_9 = g_{10} = g_{11} = g_{12}, \quad m_N = m_\Lambda = m_\Sigma = m_\Xi, \quad m_\pi = m_K, \\ d_6 = 0, \quad d_1 = -3\varepsilon d_2 = \varepsilon' d_3 = -3\varepsilon \varepsilon' d_4 = \frac{3}{2} \varepsilon'' d_5, \quad (29)$$

i.e., with $d_1 = 3$

$$j_\lambda^\psi = 3(\bar{N}\gamma_\lambda \Lambda) - \varepsilon(\bar{N}\gamma_\lambda \Sigma) + 3\varepsilon'(\bar{\Lambda}\gamma_\lambda i\tau_2 \Xi) - \varepsilon \varepsilon'(\bar{\Sigma}\gamma_\lambda i\tau_2 \tau \Xi) \\ + 2\varepsilon''(\bar{K}\tau\partial_\lambda \pi).$$

$$2) \quad g_1 = \varepsilon g_2 = g_3 = g_4, \quad g_5 = \varepsilon g_6 = -\varepsilon' g_7 = -\varepsilon \varepsilon' g_8 \\ = \varepsilon'' g_9 = -\varepsilon'' g_{10} = -\varepsilon'' g_{11} = \varepsilon'' g_{12}, \\ \varepsilon, \varepsilon', \varepsilon'' = \pm 1, \quad m_N = m_\Lambda = m_\Sigma = m_\Xi, \quad m_\rho = m_K, \\ d_5 = 0, \quad d_1 = \varepsilon d_2 = \varepsilon' d_3 = \varepsilon \varepsilon' d_4 = \varepsilon'' d_6/2, \quad (30)$$

i.e., with $d_1 = 1$

$$j_\lambda^\psi = (\bar{N}\gamma_\lambda \Lambda) + \varepsilon(\bar{N}\gamma_\lambda \Sigma) + \varepsilon'(\bar{\Lambda}\gamma_\lambda i\tau_2 \Xi) + \varepsilon \varepsilon'(\bar{\Sigma}\gamma_\lambda i\tau_2 \tau \Xi) \\ + 2\varepsilon''(\bar{K}\partial_\lambda \rho).$$

$$3) \quad g_2 = \varepsilon(g_1 - g_4)/\sqrt{3}, \quad g_3 = g_1 + g_4, \\ g_5 = \varepsilon''(2g_1 + g_4)/\sqrt{3}, \quad g_6 = \varepsilon \varepsilon'' g_4, \\ g_7 = \varepsilon' \varepsilon''(g_1 + 2g_4)/\sqrt{3}, \quad g_8 = -\varepsilon \varepsilon' \varepsilon'' g_1, \\ g_9 = \varepsilon'''(g_1 + 2g_4)/\sqrt{3}, \quad g_{10} = -\varepsilon'''(g_1 - g_4)/\sqrt{3}, \\ g_{11} = \varepsilon'''(g_1 - g_4)/\sqrt{3}, \\ g_{12} = -\varepsilon'''(2g_1 + g_4)/\sqrt{3}, \quad \varepsilon, \varepsilon', \varepsilon'', \varepsilon''' = \pm 1; \\ m_N = m_\Lambda = m_\Sigma = m_\Xi, \quad m_\rho = m_\pi = m_K, \\ d_1 = \sqrt{3} \varepsilon d_2 = \varepsilon' d_3 = -\sqrt{3} \varepsilon \varepsilon' d_4 = \sqrt{3} \varepsilon'' d_5 = \varepsilon'' \varepsilon''' d_6, \quad (31)$$

i.e., with $d_1 = 3^{1/2}$

$$j_\lambda^\psi = \sqrt{3}(\bar{N}\gamma_\lambda \Lambda) + \varepsilon(\bar{N}\gamma_\lambda \Sigma) + \sqrt{3}\varepsilon'(\bar{\Lambda}\gamma_\lambda i\tau_2 \Xi) \\ - \varepsilon \varepsilon'(\bar{\Sigma}\gamma_\lambda i\tau_2 \tau \Xi) + \varepsilon''(\bar{K}\tau\partial_\lambda \pi) + \sqrt{3}\varepsilon \varepsilon''(\bar{K}\partial_\lambda \rho).$$

$$4) \quad g_1 = g_3 = g_4 = g_6 = g_8 = 0, \\ g_2 = -\varepsilon g_5 = -\varepsilon' g_7, \quad \varepsilon, \varepsilon' = \pm 1, \\ g_9 = g_{11} = g_{12}, \quad m_N = m_\Sigma = m_\Xi, \quad m_\pi = m_K, \\ d_1 = d_3 = d_6 = 0, \quad d_5 = \varepsilon d_2 = \varepsilon' d_4, \quad (32)$$

i.e., with $d_5 = 1$

$$j_\lambda^\psi = \varepsilon(\bar{N}\gamma_\lambda \Sigma) + \varepsilon'(\bar{\Sigma}\gamma_\lambda i\tau_2 \tau \Xi) + (\bar{K}\tau\partial_\lambda \pi).$$

The last condition on the g_j is a special case of Eq. (19), and consequently, strictly speaking, it is inadmissible, because the vanishing of the divergence of the current j_λ^V is in contradiction with experiment.

In the most general case the current j_λ^χ with the isospin $3/2$ is of the form

$$(j_\lambda^\chi)_{3/2} = -h_1(\bar{\rho}\gamma_\lambda \Sigma^-) - h_2(\bar{\Sigma}^+ \gamma_\lambda \Xi^-) - h_3(\bar{K}^+ \partial_\lambda \pi^-), \\ (j_\lambda^\chi)_{1/2} = \{h_1[\sqrt{2}(\bar{\rho}\gamma_\lambda \Sigma^0) - (\bar{n}\gamma_\lambda \Sigma^-)] + h_2[\sqrt{2}(\bar{\Sigma}^0 \gamma_\lambda \Xi^-) \\ + (\bar{\Sigma}^+ \gamma_\lambda \Xi^0)] + h_3[\sqrt{2}(\bar{K}^0 \partial_\lambda \pi^0) + (\bar{K}^+ \partial_\lambda \pi^-)]\}/\sqrt{3}, \\ (j_\lambda^\chi)_{-1/2} = \{h_1[\sqrt{2}(\bar{n}\gamma_\lambda \Sigma^0) + (\bar{\rho}\gamma_\lambda \Sigma^+)] + h_2[-\sqrt{2}(\bar{\Sigma}^0 \gamma_\lambda \Xi^0) \\ + (\bar{\Sigma}^- \gamma_\lambda \Xi^-)] + h_3[\sqrt{2}(\bar{K}^0 \partial_\lambda \pi^0) + (\bar{K}^+ \partial_\lambda \pi^+)]\}/\sqrt{3}, \\ (j_\lambda^\chi)_{-3/2} = h_1(\bar{n}\gamma_\lambda \Sigma^+) - h_2(\bar{\Sigma}^- \gamma_\lambda \Xi^0) + h_3(\bar{K}^0 \partial_\lambda \pi^+). \quad (33)$$

Instead of j_λ^χ it is convenient to consider the current j_λ^φ that transforms in the isotopic space according to the reducible representation $1 \times 1/2$ and has the property that if we break it up according to the irreducible representations the component with $T = 3/2$ is equal to j_λ^χ and that with $T = 1/2$ is zero:

$$j_\lambda^\varphi = \frac{1}{3}\{h_1[2(\bar{N}\gamma_\lambda \Sigma) - i[\bar{N}\gamma_\lambda \cdot \Sigma]] + h_2[2(\bar{\Sigma}\gamma_\lambda i\tau_2 \Xi) \\ - i[\bar{\Sigma}\gamma_\lambda \cdot i\tau_2 \tau \Xi]] + h_3[\sqrt{2}(\bar{K}\partial_\lambda \pi) - i[\bar{K}\tau \cdot \partial_\lambda \pi]]\}. \quad (34)$$

If $\partial j_\lambda^\chi / \partial x_\lambda = 0$, then we also have $\partial j_\lambda^\varphi / \partial x_\lambda = 0$, and conversely. The latter equation leads to the system of equations

$$h_1 g_6 = -h_2 g_8 = -h_3 g_3 = h_3 g_4, \quad h_1 g_8 = -h_2 g_6, \\ h_2 g_3 = -h_2 g_4 = h_3 g_8, \quad h_1 g_1 = h_3 g_6 = -h_1 g_3, \quad h_1 g_6 = h_3 g_1, \\ h_2 g_2 = h_3 g_7, \quad h_2 g_7 = h_3 g_2, \quad h_1 g_2 = -h_3 g_5, \\ h_1(g_{11} - g_9) = h_2(g_{12} - h_{11}) = 0, \quad h_1(m_N - m_\Sigma) \\ = h_2(m_\Sigma - m_\Xi) = 0, \quad h_3(m_\pi - m_K) = 0. \quad (35)$$

It is easy to see that there is only one solution of the system (35) that is not forbidden by the conditions (16):

$$g_1 = -g_3 = g_4 = \tilde{\varepsilon} g_6 = \tilde{\varepsilon} \varepsilon' g_8, \quad g_2 = -\tilde{\varepsilon} g_5 = -\varepsilon' \tilde{\varepsilon} g_7, \\ \tilde{\varepsilon}_1 \varepsilon' = \pm 1, \quad g_9 = g_{11} = g_{12}, \quad m_N = m_\Sigma = m_\Xi, \quad m_\pi = m_K, \\ h_1 = -\varepsilon' h_2 = \tilde{\varepsilon} h_3. \quad (36)$$

The vanishing of the divergence of the current of Eq. (36) is compatible only with conservation of the j_λ^ψ of Eq. (29), because only in this case (and with $\tilde{\varepsilon} = \varepsilon \varepsilon'$, $g_2 = \varepsilon g_1$, $g_{10} = g_9$, $m_\Lambda = m_N$) do the conditions for the coupling constants and masses agree in the two cases. Precisely this case was pointed out in an earlier paper,^[1] where the currents with isospins $1/2$ and $3/2$ were considered together.

4. DISCUSSION OF RESULTS

In all of the arguments given above it has been assumed that the parities of Λ and Σ , and also those of N and Ξ , are the same ($P_{\Lambda\Sigma} = P_{N\Xi} = 1$), and that K and Λ have opposite parities ($P_{K\Lambda} = -1$). It is not hard to see that violation of these assumptions does not lead to the appearance

of any new conserved currents and that the "obvious" currents j_λ^S , $j_\lambda^{S_1}$, j_λ^V are conserved for any relative parities of the particles. As for the current $j_\lambda^{V_1}$ of Eq. (17), for $P_{\Lambda\Sigma} = -1$ it contains $(\bar{\Lambda}\gamma_\lambda\gamma_5\Sigma) + (\bar{\Sigma}\gamma_\lambda\gamma_5\Lambda)$ instead of $(\bar{\Lambda}\gamma_\lambda\Sigma) + (\bar{\Sigma}\gamma_\lambda\Lambda)$, and owing to this the condition $m_\Sigma - m_\Lambda = 0$ is replaced by $m_\Sigma + m_\Lambda = 0$, which is not satisfied in any approximation. Consequently, for the current $j_\lambda^{V_1}$ to be conserved we must have $P_{\Lambda\Sigma} = 1$; $P_{K\Lambda}$ and $P_{N\Xi}$ can be either $+1$ or -1 .

By similar arguments, for the conservation of $j_\lambda^{\psi_1}$, j_λ^ϕ , $j_\lambda^{\psi_3}$ we must have $P_{\Lambda\Sigma} = P_{N\Xi} = -P_{K\Lambda} = 1$ (in the last case, we must also have $P_{N\rho} = -1$). Conservation of $j_\lambda^{\psi_2}$ requires $P_{\Lambda\Sigma} = P_{N\Xi} = 1$; $P_{\Lambda K} = P_{N\rho} = \pm 1$. The solution (24) for the current j_λ^B holds for $P_{N\Xi} = 1$.

It can be seen from the formulas of Sec. 3 that the conditions for conservation of currents with change of strangeness always include equality of masses of different particles, in contradiction with experiment (it is not hard to see that owing to the symmetry of the strong interactions in such a case not only the bare masses, but also the observed masses must be equal). There are also contradictions with experiment in certain selection rules, which are consequences of symmetry properties. In particular, this is so for all cases in which strangeness is conserved, for example, the case of Eq. (16) and, as has been noted above, of the currents (18) and (19). As Pais has shown,^[4] the symmetry condition (17), which is included, in particular, in Eqs. (29) and (30), also leads to making observed reactions forbidden.

It is interesting to note that there is no such forbiddenness in the cases of Eqs. (31) and (36). Instead, one can derive a number of relations between the amplitudes for various processes. In the case of Eq. (36), for example, which allows the absence of the ρ meson, Behrends and Sirlin^[2] have established a large number of relations between the amplitudes for reactions of the type meson + nucleon $\rightarrow$ meson + baryon. Namely, they have found the following equations:

$$\begin{aligned}\Omega_1 &= \langle p\pi^+ | p\pi^+ \rangle = \langle n\pi^- | n\pi^- \rangle = \langle pK^0 | pK^0 \rangle \\ &= \langle nK^+ | nK^+ \rangle, \\ \Omega_2 &= \langle \Sigma^+ K^+ | p\pi^+ \rangle = \langle \Sigma^- K^0 | n\pi^- \rangle = -\langle nK^+ | pK^0 \rangle \\ &= -\langle pK^0 | nK^+ \rangle, \\ \Omega_3 &= \langle p\pi^- | p\pi^- \rangle = \langle n\pi^+ | n\pi^+ \rangle = \langle p\bar{K}^0 | p\bar{K}^0 \rangle \\ &= \langle nK^- | nK^- \rangle, \\ \Omega_4 &= \langle n\pi^0 | p\pi^- \rangle = -\langle p\pi^0 | n\pi^+ \rangle = \langle \Sigma^+ \pi^0 | p\bar{K}^0 \rangle \\ &= -\langle \Sigma^- \pi^0 | nK^- \rangle, \\ \Omega_5 &= \langle \Sigma^0 K^0 | p\pi^- \rangle = -\langle \Sigma^0 K^+ | n\pi^+ \rangle = -\langle \Sigma^0 \pi^+ | p\bar{K}^0 \rangle \\ &= \langle \Sigma^0 \pi^- | nK^- \rangle,\end{aligned}$$

$$\begin{aligned}\Omega_6 &= \langle \Sigma^- K^+ | p\pi^- \rangle = \langle \Sigma^+ K^0 | n\pi^+ \rangle = \langle \Xi^0 K^+ | p\bar{K}^0 \rangle \\ &= -\langle \Xi^- K^0 | nK^- \rangle, \\ \Omega_7 &= \langle pK^+ | pK^+ \rangle = \langle nK^0 | nK^0 \rangle, \quad \Omega_8 = \langle pK^- | pK^- \rangle \\ &= \langle n\bar{K}^0 | n\bar{K}^0 \rangle, \\ \Omega_9 &= \langle \Sigma^+ \pi^- | pK^- \rangle = \langle \Sigma^- \pi^+ | n\bar{K}^0 \rangle = -\langle n\bar{K}^- | pK^- \rangle \\ &= -\langle pK^- | n\bar{K}^0 \rangle, \\ \Omega_{10} &= \langle \Sigma^- \pi^+ | pK^- \rangle = \langle \Sigma^+ \pi^- | n\bar{K}^0 \rangle = \langle \Xi^0 K^0 | pK^- \rangle \\ &= -\langle \Xi^- K^+ | n\bar{K}^0 \rangle, \\ \Omega_{11} &= \langle \Sigma^0 \pi^0 | pK^- \rangle = \langle \Sigma^0 \pi^0 | n\bar{K}^0 \rangle, \\ \Omega_{12} &= \langle \Lambda K^0 | p\pi^- \rangle = \langle \Lambda K^+ | n\pi^+ \rangle = -\langle \Lambda \pi^+ | p\bar{K}^0 \rangle \\ &= -\langle \Lambda \pi^- | nK^- \rangle, \\ \Omega_{13} &= \langle \Lambda \pi^0 | pK^- \rangle = -\langle \Lambda \pi^0 | n\bar{K}^0 \rangle, \\ \Omega_{14} &= \langle \Xi^- K^+ | pK^- \rangle = -\langle \Xi^0 K^0 | n\bar{K}^0 \rangle\end{aligned}\quad (37)$$

(Behrends and Sirlin did not have Ω_{14} , and they had different signs for the third and fourth amplitudes in Ω_2 and the second amplitude in Ω_{13}), and established five additional relations between the Ω_i . It is, however, not hard to find four more relations of this kind, so that only five of the amplitudes in Eq. (37) are independent. If we regard Ω_1 , Ω_3 , Ω_7 , Ω_8 , and Ω_{12} as independent, then

$$\begin{aligned}\Omega_2 &= \Omega_1 - \Omega_7, \quad \Omega_4 = \Omega_5 = (\Omega_1 - \Omega_3)/\sqrt{2}, \quad \Omega_6 = \Omega_3 - \Omega_7, \\ \Omega_9 &= \Omega_3 - \Omega_8, \quad \Omega_{10} = \Omega_1 - \Omega_8, \quad \Omega_{11} = (\Omega_1 + \Omega_3)/2 - \Omega_8, \\ \Omega_{14} &= \Omega_7 + \Omega_8 - \Omega_1 - \Omega_3, \quad \Omega_{13} = -\Omega_{12}/\sqrt{2}.\end{aligned}\quad (38)$$

All of these relations are true if in Eq. (36) $\epsilon' = \tilde{\epsilon} = 1$ (or if in the formula IV of Behrends and Sirlin's paper^[2] $\eta_1 = \eta_2 = 1$). But as before they are correct if in the definitions of the Ω_i we replace Σ by $\tilde{\Sigma}$ and Ξ by $\epsilon'\Xi$.

Analogous equations can also be obtained in the case of Eq. (31). It may be remarked that in a recent preprint by Gell-Mann^[12] the symmetry of this same case (31) is analyzed with the additional condition $g_1 = g_4$.

In the low-energy region it is hard to give relations of the type of Eqs. (37) and (38) any physical meaning, if only for the reason that because of the difference of the masses it is not clear even at what energies one is to compare the amplitudes of the various processes in these equations. In the high-energy case, on the other hand, when the mass difference is not so important, it would be interesting to get comparisons with experiment for relations of this type, and of course also for the selection rules of Pais.

5. THE MODEL OF SAKATA AND OKUN'

If we assume that all baryons, π mesons, and K mesons consist of p , n , Λ , $\bar{p}$, $\bar{n}$, and $\bar{\Lambda}$,^[5,6] then in this model the most general four-fermion

(strong) interaction between the basic particles is of the form

$$L = \frac{1}{2} \sum_j f_j^{(1)} (\bar{N} O_j) (\bar{N} O_j) + \frac{1}{2} \sum_j f_j^{(2)} (\bar{\Lambda} O_j \Lambda) (\bar{\Lambda} O_j \Lambda) + \sum_j f_j^{(3)} (\bar{N} O_j N O_j) (\bar{\Lambda} \Lambda), \quad (39)$$

where only the antisymmetric interaction types contribute to the first two sums. Arguments like those that have been given show that for the Lagrangian (39), with arbitrary values of the $f_j^{(i)}$, currents with zero divergence are

$$j_\lambda^S = (\bar{N} \gamma_\lambda N) + (\bar{\Lambda} \gamma_\lambda \Lambda), \quad j_\lambda^{S_1} = -(\bar{\Lambda} \gamma_\lambda \Lambda), \quad (40)$$

$$j_\lambda^V = (\bar{N} \gamma_\lambda \tau N).$$

Moreover, as has already been pointed out by Okun',^[7] in this model it is not possible to construct either a nonconserved isovector current or a current with vanishing divergence that does not have an electromagnetic analog. In the language of Sec. 2, the Sakata model forbids isovector currents which are not the same as the j_λ^V of Eq. (15).

As for the only isospinor current,

$$j_\lambda^\psi = (\bar{N} \gamma_\lambda \Lambda), \quad (41)$$

we have

$$\frac{\partial j_\lambda^\psi}{\partial x_\lambda} = (m_N - m_\Lambda) (\bar{N} \Lambda) - \sum_j f_j^{(1)} (\bar{N} O_j \Lambda) (\bar{N} O_j N) + \sum_j f_j^{(2)} (\bar{N} O_j \Lambda) (\bar{\Lambda} O_j \Lambda) + \sum_j f_j^{(3)} [(\bar{N} O_j \Lambda) (\bar{N} O_j N) - (\bar{N} O_j \Lambda) (\bar{\Lambda} O_j \Lambda)], \quad (42)$$

and the condition $\partial j_\lambda^\psi / \partial x_\lambda = 0$ is satisfied only in the case in which $f_j^{(3)} = 0$ for the symmetric types, and for the antisymmetric

$$f_j^{(1)} = f_j^{(2)} = f_j^{(3)}, \quad m_N = m_\Lambda. \quad (43)$$

The condition (43) means absolute identity of N and Λ , and strictly speaking is incorrect.^[13] Analogs of Eq. (43) are found in Eqs. (29) and (30), where there is complete degeneracy of the baryons.

APPENDIX

THE ISOVECTOR CURRENTS (THE CASE OF EQ. (16))

The solution of (13) with $b_5 = 0$ has meaning only for $b_7 = 0$. Otherwise $g_9 = g_{10} = g_{11} = g_{12}$, i.e., the ρ mesons do not interact with baryons at all, and this makes it altogether senseless to introduce the term with $b_7 \neq 0$. For the same reason we must suppose that $b_1 = 0$ for $g_1 = g_5 = g_6 = 0$, b_3

$= 0$ for $g_2 = g_3 = g_6 = g_8 = 0$, $b_4 = 0$ for $g_4 = g_7 = g_8 = 0$, $b_5 = 0$ for $g_1 = g_2 = g_3 = g_4 = 0$, and $b_6 = 0$ for $g_5 = g_6 = g_7 = g_8 = 0$. This also applies to the same extent to the currents (14) and (17)–(19), where all such b_i must be stricken out, providing only (in the last three cases) that the vanishing of the corresponding constants g_j is not in contradiction with the conditions on the coupling constants contained in (17)–(19).

If now we write out again the currents obtained from Eqs. (14) and (17)–(19) by all possible "striking out," then according to Eq. (13) with $b_5 = 0$ the divergence of the vector current vanishes only when one or another of the following systems of equations is satisfied:

- 1) $g_1 = g_6 = g_7 = g_8 = 0$, $q_4, g_5 \neq 0$,
 $b_8 = b_2 = b_3 = b_4 = 0$, $b_1 = b_6$;
- 2) $g_1 = g_2 = g_3 = g_7 = g_8 = 0$, $g_4, g_6 \neq 0$,
 $b_8 = b_2 = b_4 = 0$, $b_1 = b_6 = b_3/2$;
- 3) $g_7 = g_8 = 0$, $g_5 = \varepsilon g_6 \neq 0$, $\varepsilon = \pm 1$, $g_{10} = g_{11}$,
 $m_\Sigma = m_\Lambda$, $b_8 = b_4 = 0$;
- a) $g_2 = \varepsilon g_3$, $b_1 = 0$, $b_6 = b_3 = -\varepsilon b_2$;
- b) $g_2 = -\varepsilon g_3$, $g_1 = 0$, $b_6 = 0$, $b_1 = b_3 = \varepsilon b_2$,

and also in the cases differing from those enumerated by the replacement

$$g_1 \leftrightarrow g_4, \quad g_5 \leftrightarrow g_7, \quad g_6 \leftrightarrow g_8, \quad b_1 \leftrightarrow b_4.$$

¹V. M. Shekhter, JETP **36**, 581 (1958), Soviet Phys. JETP **9**, 403 (1959).

²R. E. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961).

³M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).

⁴A. Pais, Phys. Rev. **110**, 574 (1958).

⁵S. Sakata, Prog. Theoret. Phys. **16**, 686 (1956).

⁶L. B. Okun', JETP **34**, 469 (1958), Soviet Phys. JETP **7**, 322 (1958).

⁷L. B. Okun', Proc. 1958 Ann. Int. Conf. on High Energy Physics at CERN, Geneva, 1958, page 223.

⁸R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

⁹L. Alvarez, Proc. Int. Conf. on High Energy Particle Physics, Kiev, 1959, AN SSSR, 1960.

¹⁰I. Yu. Kobzarev and L. B. Okun', JETP **39**, 210 (1960), Soviet Phys. JETP **12**, 150 (1961).

¹¹G. Feinberg and R. E. Behrends, Phys. Rev. **115**, 745 (1959).

¹²M. Gell-Mann, Preprint.

¹³W. Thirring, Nuclear Phys. **10**, 97 (1959).

Translated by W. H. Furry

THE ANALYTIC PROPERTIES OF THE TOTAL π N INTERACTION CROSS SECTION AS A FUNCTION OF VIRTUALITY

I. M. DREMIN

Submitted to JETP editor March 25, 1961

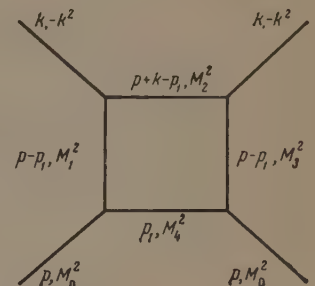
J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 821-827 (September, 1961)

The analytic properties of the imaginary part of the forward amplitude for elastic scattering of π mesons on nucleons as a function of virtuality (i.e., the square of the four-momentum) of the π meson are considered in perturbation theory as well as on the basis of the Jost-Lehmann-Dyson spectral representation. In virtue of the optical theorem such an analysis yields information on the analytic properties of the total π N interaction cross section as a function of virtuality. The role of the "anomalous" thresholds is discussed. The possibilities for obtaining the total cross section as a multiplicative function of the total energy in the center-of-mass system and of the virtuality are discussed.

1. A number of recent papers have been devoted to the investigation of inelastic processes at high energies in the pole approximation (see, for example, [1-7]). The assumptions restricting the pole approximation have been enumerated earlier. [4] Clearly it is highly desirable to go beyond the limitations of the pole approximation. For this purpose it is necessary, in particular, to obtain some information on the behavior of the total cross section as a function of virtuality.* A first step in this direction is the investigation of the analytic properties of the total cross section as a function of virtuality for a fixed value of the energy in the center-of-mass system. In virtue of the optical theorem, which relates the imaginary part of the forward scattering amplitude to the total cross section, the analytic properties of the total cross section are given by the analytic properties of the imaginary part of the forward scattering amplitude. With this in mind, we now turn to the consideration of the analytic properties of the imaginary part of the forward amplitude for elastic π N scattering as a function of the virtuality of the π meson.

2. Let us first determine the analytic properties of the imaginary part of the forward elastic scattering amplitude in the k^2 plane, using fourth order perturbation theory. The corresponding diagram is shown in Fig. 1. The squares of the masses of each of the particles and their four-

FIG. 1. Fourth order diagram in perturbation theory.



momenta are indicated. To keep the considerations general, we shall not as yet assume that $M_0 = M_1 = M_2 = M_3 = m$ and $M_4 = \mu$ (m is the nucleon mass and μ is the pion mass). The imaginary part of the amplitude for this process is written in the form

$$\text{Im } f_4(0^0) = \frac{g^4}{8\pi^2} \quad (1)$$

$$\times \int \frac{d^4 p_1 \delta((p+k-p_1)^2 + M_2^2) \delta(p_1^2 + M_4^2) \theta(k_0 - p_0 - p_{10}) \theta(p_{10})}{[(p-p_1)^2 + M_1^2][(p-p_1)^2 + M_3^2]}.$$

Evaluating the integral on the right hand side of (1) and introducing the notation

$$s = -(p+k)^2, \quad \tau = -k^2/s, \quad \beta_i^2 = M_i^2/s, \quad (2)$$

we find

$$\text{Im } f_4(0^0) = \frac{g^4}{32\pi s^2} \frac{[(1 + \beta_4^2 - \beta_2^2)^2 - 4\beta_4^2]^{1/2}}{\beta_1^2 - \beta_3^2} \times \ln \frac{(a_1 + Q^{1/2})(a_3 - Q^{1/2})}{(a_1 - Q^{1/2})(a_3 + Q^{1/2})}, \quad (3)$$

where

$$a_i = 2(\beta_0^2 + \beta_4^2 - \beta_i^2) - (1 + \beta_0^2 - \tau)(1 + \beta_4^2 - \beta_i^2) \quad (i = 1, 3),$$

$$Q = [(1 + \beta_0^2 - \tau)^2 - 4\beta_0^2][(1 + \beta_4^2 - \beta_2^2)^2 - 4\beta_4^2].$$

*The term "virtuality" is used to describe the square of the four-momentum of one of the incoming particles. We remark that one is most interested in the behavior of the total cross section as a function of virtuality on the real half-axis $\text{Re } k^2 > 0$ (the metric is chosen such that $k^2 = k^2 - k_0^2$). [4]

The analysis of this expression is considerably simplified if we assume that $M_1 = M_3$; this assumption does not diminish the generality of our discussion in an essential manner. Then formula (3) takes the form

$$\operatorname{Im} f_4(0^\circ) = \frac{g^4}{16\pi s^2} \frac{[(1 + \beta_4^2 - \beta_2^2)^2 - 4\beta_4^2]^{1/2}}{(\tau - \tau_3)(\tau - \tau_4)},$$

$$\tau_{3,4} = [\beta_4^2 + \beta_1^2 - \beta_0^2 + \beta_0^2(\beta_2^2 + \beta_4^2) - (\beta_1^2 - \beta_4^2)(\beta_2^2 - \beta_4^2) \pm iP^{1/2}/2\beta_4^2,$$

$$P = ((1 + \beta_4^2 - \beta_2^2)^2 - 4\beta_4^2)((\beta_0 + \beta_4)^2 - \beta_1^2)(\beta_1^2 - (\beta_0 - \beta_4)^2). \quad (4)$$

It is seen from expression (4) that the imaginary part of the forward elastic scattering amplitude has two poles in the complex τ plane. The position of these poles in the k^2 plane (s is assumed fixed and real) can be easily obtained with the help of (2) and (4).

We see that the poles are situated symmetrically with respect to the real k^2 axis. At the reaction threshold [i.e., for $s = (M_2 + M_4)^2$] they coalesce into a single point on the real axis, and with increasing s they move away from the real axis along a hyperbola in the k^2 plane with the asymptotes

$$\operatorname{Im} k^2 = \pm \frac{[(M_0 + M_4)^2 - M_1^2]^{1/2} [M_1^2 - (M_0 - M_4)^2]^{1/2}}{M_1^2 + M_4^2 - M_0^2} \times [\operatorname{Re} k^2 + M_1^2 + M_2^2].$$

It is interesting to note that for $M_1^2 + M_4^2 - M_0^2 > 0$ (this case corresponds to the so-called "normal" thresholds^[8]) these poles will always move away from the region of interest $\operatorname{Re} k^2 > 0$ as the energy s increases.^[4] On the other hand, if $M_1^2 + M_4^2 - M_0^2 < 0$ (this corresponds to the "anomalous" thresholds^[8]), the poles will move into the region $\operatorname{Re} k^2 > 0$ as the energy increases beyond a certain value. The distance from these poles to the region of interest can be of the same order of magnitude as the distance from this region to the point $k^2 = -\mu^2$ corresponding to the pole of the propagation function of the π meson.

We shall not discuss this problem in any detail, since the presence of poles in the complex k^2 plane in perturbation theory and their role in the extrapolation procedure proposed by Chew and Low^[1] have already been discussed by Ascoli.^[9] We emphasize only that the above discussion shows that these poles correspond to poles of the total cross section as a function of the virtuality. For an illustration we give the location of the poles of the πN scattering cross section as a function of

virtuality in fourth order perturbation theory (i.e., for $M_0 = M_1 = M_2 = M_3 = m$, $M_4 = \mu$). These poles are situated at the following points of the k^2 plane:

$$k_{3,4}^2 = -\frac{1}{2}\{s + 3m^2 - \mu^2 \pm i(4m^2/\mu^2 - 1)^{1/2}[(s - m^2 + \mu^2)^2 - 4s\mu^2]^{1/2}\}.$$

At the reaction threshold $s = (m + \mu)^2$ we have

$$k_3^2 = k_4^2 = -m(2m + \mu) \ll -\mu^2.$$

3. Let us now consider the analytic properties of the total cross section in the k^2 plane on the basis of the spectral representation of Jost, Lehmann, and Dyson.^[10] It was shown in the paper of Lehmann^[10] [formula (23)] that the imaginary part of the forward elastic scattering amplitude as a function of s and k^2 can be written in the form [with $s \geq (m + \mu)^2$ in the case of πN scattering]

$$\begin{aligned} \operatorname{Im} T(0^\circ) &= \frac{1}{16\pi K_0^2} \int du_{0i} u_i du_i d\kappa_{0i}^2 \\ &\times \int_0^{2\pi} d\alpha \int_0^{2\pi} d\chi \int_{-1}^1 d(\cos \varphi_1) \int_{-1}^1 d(\cos \varphi_2) \\ &\times \frac{\Phi(u_{0i}, u_i^2, \kappa_{0i}^2, \cos \alpha \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2, s)}{[X_1 - \sin \varphi_1 \cos \chi][X_2 - \sin \varphi_2 \cos(\chi - \alpha)]}, \end{aligned} \quad (5)$$

where

$$X_i = \{K_0^2 + u_i^2 + \kappa_{0i}^2 - [(m^2 + k^2)/2s^{1/2} + u_{0i}]^2\}/2K_0 u_i \quad (i = 1, 2),$$

$$K_0^2 = [(s + m^2 + k^2)^2 - 4m^2 s]/4s,$$

and the function Φ is arbitrary in the region

$$\begin{aligned} 0 \leq u_i \leq s^{1/2}/2, \quad -s^{1/2}/2 + u_i \leq u_{0i} \leq s^{1/2}/2 - u_i, \\ \kappa_{0i} \geq \max\{0; m_1 - \sqrt{(s^{1/2}/2 + u_{0i})^2 - u_i^2}, \\ m_2 - \sqrt{(s^{1/2}/2 - u_{0i})^2 - u_i^2}\} \end{aligned} \quad (6)$$

($m_1 = 3\mu$ and $m_2 = m + \mu$ in the case of πN scattering); outside this region the function Φ vanishes. From (5) we can derive the analytic properties with respect to the virtuality k^2 . Indeed, singularities can occur only at points k^2 for which the denominator in (5) vanishes, since the function Φ is independent of k^2 . If the denominator is to vanish, at least one of the conditions

$$-1 \leq X_1 \leq 1, \quad -1 \leq X_2 \leq 1. \quad (7)$$

must be satisfied. These conditions are identical in the sense that they define the same region of analyticity in the k^2 plane owing to the symmetry of the conditions (6) with respect to the index i .

We shall therefore leave out this index in the following.

Using formula (6), we can express the virtuality k^2 as a function of X , u , u_0 , κ_0 , and s . In addition to the variables (2) it is convenient to introduce also the variables

$$x = \left[\left(\frac{1}{2} + \frac{u_0}{s^{1/2}} \right)^2 - \frac{u^2}{s} \right]^{1/2},$$

$$y = \left[\left(\frac{1}{2} - \frac{u_0}{s^{1/2}} \right)^2 - \frac{u^2}{s} \right]^{1/2},$$

$$\kappa^2 = \kappa_0^2/s, \quad z = 1 - X^2, \quad \beta^2 = m^2/s. \quad (8)$$

Then

$$\begin{aligned} \tau &= 1 + \beta^2 - \frac{2(y^2 + \beta^2 - \kappa^2)(1 - x^2 + y^2)}{4y^2 + (1 - (x + y)^2)(1 - (x - y)^2)z} \\ &\pm \frac{2\sqrt{1 - z}[(1 - (x + y)^2)(1 - (x - y)^2)]^{1/2}}{4y^2 + [1 - (x + y)^2][1 - (x - y)^2]z} \\ &\times \{(y^2 + \beta^2 - \kappa^2)^2 - 4\beta^2 y^2 \\ &- [1 - (x + y)^2][1 - (x - y)^2]\beta^2 z\}^{1/2}. \end{aligned} \quad (9)$$

As we have shown above [formulas (6) and (7)], the imaginary part of the forward scattering amplitude can have singularities only in the following region of the variables:

$$\begin{aligned} x &\geq 0, \quad y \geq 0, \quad x + y \leq 1, \\ \kappa &\geq \max\{0, \beta_1 - x; \beta_2 - y\}, \quad 0 \leq z \leq 1. \end{aligned} \quad (10)$$

The expression under the root sign in (9) can be positive or negative. If it is negative, the singularities will appear in the complex k^2 plane. Then

$$\tau = \tau_1 \pm i\tau_2, \quad (11)$$

where

$$\begin{aligned} \tau_1 &= 1 + \beta^2 - \frac{2(y^2 + \beta^2 - \kappa^2)(1 - x^2 + y^2)}{4y^2 + [1 - (x + y)^2][1 - (x - y)^2]z} \\ &\equiv 1 + \beta^2 - 2b, \\ \tau_2^2 &= 4(1 - z)[1 - (x + y)^2][1 - (x - y)^2] \\ &\times \{\beta^2(4y^2 + [1 - (x + y)^2]) \\ &\times [1 - (x - y)^2]z - (y^2 + \beta^2 - \kappa^2)^2\} \\ &\times \{4y^2 + [1 - (x + y)^2][1 - (x - y)^2]z\}^{-2}. \end{aligned}$$

It is seen from (11) that the singularities are situated symmetrically with respect to the real τ axis.

In order to determine the boundaries of the region of analyticity, we require first of all the minimal value of τ_1 for which the variables x , y , κ , and z are in the region (10). An elementary

calculation yields

$$\begin{aligned} \tau_{1min} &= 1 - \sqrt{3}\beta_2/y_0 \\ &+ 9(\beta_2^2 - \beta^2)/8y_0^2 + (\beta_2^2 + \beta^2)/2, \end{aligned} \quad (12)$$

where

$$y_0 = \cos \frac{\pi + \varphi}{3}, \quad \cos \varphi = \frac{3\sqrt{3}\beta_2^2 - \beta^2}{\beta_2}.$$

Hence the imaginary part of the forward elastic scattering amplitude cannot have singularities for $\tau_1 < \tau_{1min}$ if $\tau_2^2 > 0$.

Expression (12) is simplified considerably in the case of πN scattering with $s \geq (m + \mu)^2$. Noting that in this case $(\beta_2^2 - \beta^2)/\beta_2 \ll 1$, we obtain

$$\tau_{1min} \approx -(2\beta^2 - \beta_2^2)/2(\beta_2^2 - \beta^2) + (3\beta^2 - \beta_2^2)/2s. \quad (13)$$

From this we find

$$\begin{aligned} (\text{Re } k^2)_{max} \\ \approx s(2m^2 - m_2^2)/2(m_2^2 - m^2) - \frac{1}{2}(3m^2 - m_2^2), \end{aligned} \quad (14)$$

i.e., asymptotically (for $s \rightarrow \infty$) the boundary of the region of analyticity in the k^2 plane moves toward the region of positive values of $\text{Re } k^2$, the shift being proportional to s .

Let us now determine the maximal and minimal values of τ_2^2 for a given fixed τ_1 . We notice, first of all, that

$$\begin{aligned} \tau_2^2 &= 4(b - r)(\beta^2/r - b), \\ r &= (y^2 + \beta^2 - \kappa^2)/(1 - x^2 + y^2). \end{aligned} \quad (15)$$

Thus, for a given τ_1 (i.e., for a given b), τ_2^2 is a function of the single variable r . The signs of b and r are identical and are determined by the sign of the expression $y^2 + \beta^2 - \kappa^2$.

It is seen from (15) that it is sufficient for finding the maximal value of τ_2^2 for given b to determine the minimal value of r for the same b [taking account of the restrictions on the region of the variables (10)] and substitute it in (15). In this way we obtain for $b \geq 0$

$$\frac{\tau_{2max}^2}{(1 - q^2)^2} + \frac{(1 + \beta^2 - \tau_1)^2}{(1 + q^2)^2} = \frac{\beta^2}{q^2}, \quad (16)$$

where $q = (\beta_2^2 - \beta^2)/2\beta_2$. For πN scattering and $s \geq (m + \mu)^2$, we have $q \ll 1$, and (16) takes the form

$$\tau_{2max}^2 + (1 + \beta^2 - \tau_1)^2 = 4\beta_2^2\beta^2/(\beta_2^2 - \beta^2)^2. \quad (17)$$

Thus, for $\tau_{1min} \leq \tau_1 \leq 1 + \beta^2$, singular points can only be found inside the ellipse described by Eq. (16).

To find the minimal value of τ_2^2 for given τ_1 , we must determine the maximal value of r for

given b . The resulting expressions are very complicated and are, therefore, given in the Appendix. Here we quote only the expression for the minimal value of τ_2^2 as a function of τ_1 , with

$$\tau_{2min}^2 = (1 - \tau_1)^2 \left(\frac{2\beta^2}{\beta_2^2 - (1 - \tau_1)(\beta_2^2 - \beta^2) + \beta_2 [\beta_2^2 - 2(1 - \tau_1)(\beta_2^2 - \beta^2)]^{1/2}} - 1 \right). \quad (19)$$

Equation (19) defines the boundary of the region of analyticity closest to the real axis. It is easily seen from (17) and (19) that the boundaries of the region of analyticity in the k^2 plane move asymptotically (for $s \rightarrow \infty$) away from the real axis, the shift being proportional to s .

The shape of the region of analyticity in the k^2 plane for πN scattering at the reaction threshold [$s = (m + \mu)^2$] is shown in Fig. 2 [condition (18) is not satisfied here, and we must therefore use the formulas given in the Appendix]. It is interesting to note that at the reaction threshold the cross section has singularities on the real axis close to the pole of the π meson propagation function at $k^2 = \mu^2$. Indeed, the closest singularity (point A in Figs. 2 and 3) for $s = (m + \mu)^2$ is situated at the point $k^2 = -3\mu^2(2m + \mu)/(2m - \mu)$.

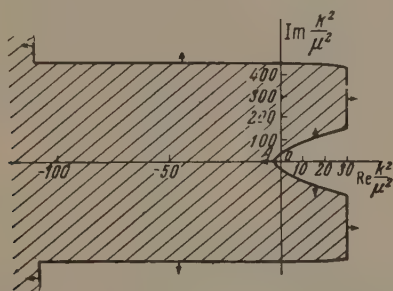


FIG. 2. Region of analyticity of the total πN interaction cross section in the k^2 plane with $s = (m + \mu)^2$ (not shaded). The scale of the imaginary axis is chosen ten times smaller than that of the real axis.

The directions in which the boundaries of the region of analyticity and the point A move as s is increased are indicated in Fig. 2 by arrows. Figure 3 shows in a larger scale the shape of the boundary of the region of analyticity near the point $k^2 = -\mu^2$ for $s = (m + \mu)^2$. For all s satisfying condition (18) [for πN scattering this means $s \geq (m + 3\mu)^2$] the shape of the boundary near the point $k^2 = -\mu^2$ changes (see Fig. 4), while the other boundaries keep their previous shape, as shown in Fig. 2. The boundary point $\text{Re } k^2 = -(3\mu)^2$ is not shifted any more as the energy changes. The other boundary points move along the directions indicated by the arrows as the energy s is increased. As already noted, this shift is asymptotically proportional to s .

$\tau_{1min} \leq \tau_1 \leq \beta_1^2$ and the following restrictions on s :

$$s \gg (m_2 - m_1)^2, \quad s \gg (m + m_1)^2. \quad (18)$$

In this case we have

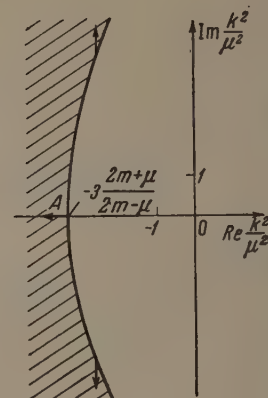


FIG. 3. Shape of the boundaries of the region of analyticity near the point $k^2 = -\mu^2$ for $s = (m + \mu)^2$.

4. From the results obtained above we can draw a number of conclusions. First of all, we note the following circumstance. We found in fourth order perturbation theory that the singularities of the πN interaction cross section as functions of the virtuality k^2 can occur only in the region $\text{Re } k^2 < 0$ and are considerably farther removed from the region of interest $\text{Re } k^2 > 0$ (see the footnote above) than the pole at $k^2 = -\mu^2$. The spectral representation then indicates the possibility that singularities may also appear in the region $\text{Re } k^2 > 0$ of the complex k^2 plane; near the reaction threshold these singularities may be situated at a distance from the region of interest which is comparable with the distance from the pole of the π meson propagation function to that region.

The occurrence of these singularities may have a great effect on the possibility of carrying out the extrapolation procedure proposed by Chew and Low^[4] not only in the "anomalous" cases but also in the case of an inelastic NN interaction with formation of a single π meson. Since, however,

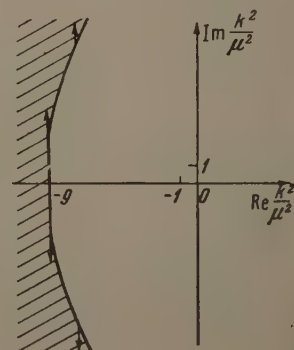


FIG. 4. Shape of the boundaries of the region of analyticity near the point $k^2 = -\mu^2$ for $s > (m + 3\mu)^2$.

the singularities in the complex plane move away from the real axis as s is increased, it appears to be probable that their effect will become smaller as the energy increases.

Furthermore, it follows immediately from the analytic properties of the total cross section as a function of the virtuality, as obtained above, that the total cross section cannot be a multiplicative function of s and k^2 for arbitrary values of these variables. Indeed, if the function $\sigma(s, k^2)$ were multiplicative, $\sigma(s, k^2) = f_1(s) f_2(k^2)$, this would imply that the position of the poles in the k^2 plane is independent of the value of s . We have shown above that this is not the case either in perturbation theory or on the basis of the spectral representation.

However, sometimes one is interested in a multiplicative total cross section on the real k^2 axis for $0 < k^2 \lesssim \mu^2$ (this problem has been discussed earlier^[5]). The requirement that the total cross section be multiplicative on a small segment of the real axis and at the same time have the analytic properties obtained above imposes certain restrictions on the form of the total cross section as a function of s and k^2 . This problem will be investigated separately.

Ascoli^[11] has recently obtained the analytic properties of the amplitude for processes with an arbitrary number of outgoing lines with respect to two transferred momenta. If, in particular, we consider such amplitudes in the one-meson approximation, we expect that the region of their analyticity with respect to one of the transferred momenta, the other one remaining fixed, is at most as large as the region of analyticity of the cross section as a function of the virtuality (the virtuality of the π meson coincides here with the transferred momentum). The region of analyticity that we have obtained is considerably larger than the region of analyticity in the work of Ascoli,^[11] since we are interested only in the one-meson term in the processes considered by Ascoli.

In conclusion I take this opportunity to thank D. S. Chernavskii for his steadfast interest in this work and for a discussion of the results.

APPENDIX

We give here the formulas for the minimal value of τ_2^2 for given τ_1 .

1) For $b \geq b_0 = [1 + (\beta_2 - \beta_1)^2] [\beta_2^2 + \beta^2 - 2\beta_2\beta_1] \times 4^{-1}(\beta - \beta_1)^{-2}$:

$$\tau_{2min}^2 = 4 [b - R_0] \left[\frac{\beta^2}{R_0} - b \right];$$

$$R_0 = \frac{2\beta_2(y_1 - q)}{1 + y_1^2}, \quad y_1 = \frac{p}{3} - \frac{2}{3} |p^2 - 3|^{1/2} \cos \frac{\pi + \varphi}{3},$$

$$\cos \varphi = \frac{|2p^3 - 9p + 27q|}{2|p^2 - 3|^{3/2}}, \quad p = \frac{2b}{\beta_2} + q. \quad (A.1)$$

2) For $b \leq b_0$:

$$\tau_{2min}^2 = 4 [b - R_1] [\beta^2 / R_1 - b];$$

$$R_1 = 2\beta_2(y_2 - q) / [1 - (\beta_2 - \beta_1)^2 + 2(\beta_2 - \beta_1)y_2],$$

$$y_2 = \frac{1}{4} \{ \beta_2 [1 - (\beta_2 - \beta_1)^2] - (\beta_2 - \beta_1)(\beta_2^2 - \beta^2) + [(\beta_2 [1 - (\beta_2 - \beta_1)^2] - (\beta_2 - \beta_1)(\beta_2^2 - \beta^2)]^2 - 4b(\beta_2^2 - \beta^2) \times [1 - (\beta_2 - \beta_1)^2]^{1/2} \} [b - \beta_2(\beta_2 - \beta_1)]^{-1}. \quad (A.2)$$

If conditions (18) are satisfied, (A.1) and (A.2) go over into formula (19).

3) If s does not satisfy condition (18), the coordinate of the point A (see Fig. 2) is determined by the expression

$$k_A^2 = -[s\sqrt{s} - (m_2 + m_1)s - (m^2 - 2m_1m_2)\sqrt{s} + m^2(m_2 - m_1)] \times (\sqrt{s} + m_2 - m_1)^{-1}. \quad (A.3)$$

Formula (A.3) agrees with formula (4.5) of the paper of Todorov,^[12] in which the boundaries of the region of analyticity on the real k^2 axis were determined.

¹G. F. Chew and F. E. Low, Phys. Rev. 113, 1640 (1959).

²V. B. Berestetskii and I. Ya. Pomeranchuk, JETP 39, 1078 (1960), Soviet Phys. JETP 12, 752 (1960).

³I. M. Dremin and D. S. Chernavskii, JETP 38, 229 (1960), Soviet Phys. JETP 11, 167 (1960).

⁴G. Gramenitskii, Dremin, Maksimenko, and Chernavskii, JETP 40, 1093 (1961), Soviet Phys. 13, 771 (1961).

⁵I. M. Dremin and D. S. Chernavskii, JETP 40, 1333 (1961), Soviet Phys. JETP.

⁶F. Salzman and G. Salzman, Phys. Rev. Lett. 5, 377 (1960).

⁷S. D. Drell, Phys. Rev. Lett. 5, 342 (1960).

⁸Karplus, Sommerfield, and Wichmann, Phys. Rev. 111, 1187 (1958).

⁹R. Ascoli, Nuovo cimento 18, 744 (1960).

¹⁰H. Lehmann, Nuovo cimento 10, 579 (1958).

¹¹R. Ascoli, Nuovo cimento 18, 754 (1960).

¹²I. T. Todorov, Nucl. Phys. 18, 521 (1960).

SECOND SOUND, THE CONVECTIVE HEAT TRANSFER MECHANISM, AND EXCITON EXCITATIONS IN SUPERCONDUCTORS

V. L. GINZBURG

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor March 28, 1961; resubmitted May 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 828-834 (September, 1961)

The possibility of observing convective heat transfer and second sound in superconductors is discussed. The change in the single-particle excitation spectrum and, in particular, the existence in superconductors of exciton excitations can in principle greatly reduce the role played by impurity scattering. As a result, the attenuation of second sound may be substantially diminished, while the convective transfer of heat may increase markedly. Some experimental data relative to superconducting lead are discussed. A mechanism is indicated which may lead to a Knight frequency shift in superconductors in the case of small samples and strong magnetic fields.

THE possibility of the simultaneous existence in superconductors of superconducting and normal currents with densities j_s and j_n should lead to the appearance of convective thermal transport in a non-uniformly heated metal, particularly in the absence of a total current $j = j_s + j_n$ (see [1]). A number of estimates [2-4] have led, however, to the conclusion that this effect is insignificant. This is explained, in the first place, by the strong friction opposing the movement of the normal component of the electron fluid, which is due fundamentally to the presence of a residual resistance. On this basis, one may conclude that the propagation of weakly-damped second sound in superconductors is impossible.

Actually, within the framework of two-fluid hydrodynamic theory, by analogy with the case of superfluidity [5] we obtain directly, in the linear approximation, the equations

$$\begin{aligned} \frac{\partial}{\partial t} \frac{m}{e} j / \partial t &= -\nabla p - \rho_{n0} v v_n + \frac{e}{m} \rho_0 E, & \frac{\partial \rho}{\partial t} + \operatorname{div} \frac{m}{e} j &= 0, \\ \frac{\partial v_s}{\partial t} &= -\nabla \mu + \frac{e}{m} E, & \frac{\partial S}{\partial t} + S_0 \operatorname{div} v_n &= 0, \\ \operatorname{div} E &= \frac{4\pi e}{m\epsilon} (\rho - \rho_0), & dp &= S_0 dT + \rho_0 d\mu, \\ \frac{m}{e} j &= \rho_s v_s + \rho_n v_n, & j_s &= \frac{e}{m} \rho_s v_s, & j_n &= \frac{e}{m} \rho_n v_n. \end{aligned} \quad (1)$$

Here the zero subscript refers to the ground-state values, and is omitted hereafter. The momentum current density is taken to be $(m/e)j$, where e/m is the free electron charge-to-mass ratio; ϵ is the dielectric constant, which is not connected with the motion under consideration, and ν is a certain effective number of collisions.

If we discount the possibility that the value of ϵ may be anomalously large, [6] which in any case lies beyond the bounds of the model used, the plasma frequency $\omega_0 = \sqrt{4\pi e^2 \rho_0 / m^2 \epsilon}$ turns out to be extraordinarily high. The system (1) for the propagation of low-frequency waves proportional to $\exp i(\omega t - k \cdot r)$ leads therefore to the relation

$$k^2 = \omega^2 \left(1 - i \frac{\rho_s}{\rho} \frac{\nu}{\omega} \right) \times \frac{\rho_n \rho [(\partial S / \partial T)_p (\partial \rho / \partial p)_T - (\partial S / \partial p)_T (\partial \rho / \partial T)_p]}{\rho_s S^2 (\partial \rho / \partial p)_T}. \quad (2)$$

We obtain from this, for $\nu = 0$ and $(\partial S / \partial T)(\partial \rho / \partial p) \gg (\partial S / \partial p) \cdot (\partial \rho / \partial T)$, the well-known formula for the second sound velocity in He II:

$$u_2 = \sqrt{\rho_s S^2 T / \rho \rho_n C}, \quad (3)$$

where the entropy S and the specific heat $C = T \partial S / \partial T$ are referred to unit volume, ρ is the total density, of the electron fluid, and ρ_s and ρ_n are the respective densities of the superfluid and normal components of the fluid.

If for normal motion in superconductors ν is of the same order as in the non-superconducting case, then there can be no possibility for propagation of low-frequency second sound.* Actually, the attenuation in (2) will be weak provided $\nu \sim v_0 / l_s \ll \omega$ ($v_0 \sim 10^8$ is the velocity at the Fermi boundary,

*We note that the extremely strong attenuation of second sound in superconductors has been emphasized in a number of papers. [7,8] I. M. Khalatnikov and the author arrived at the same conclusion, independently of one another, some two years ago.

while l_s is the mean free path for scattering by impurities). This gives $l_s \gg v_0/\omega \sim 10^8/\omega$, and even for $\omega \sim 10^8$ the extremely stringent condition $l_s \gg 1$ cm must be fulfilled.

Even this condition, however, is insufficient if the inequality $\lambda = 2\pi u_2/\omega \gg l_0$, required for the application of hydrodynamic theory, is not satisfied. The effective excitation mean free path l_0 necessary for establishment of hydrodynamic flow is of the same order as the mean free path l_e for electron-electron collisions only when $l_e \ll l_s$. Further, for a normal metal $l_e \sim (1/\sigma_e n_0) \times (\epsilon_0/kT)^2 \sim 3 \times 10^{-3} (100/T)^2$, where $\sigma_e \sim 10^{-15}$, $n_0 \sim 3 \times 10^{22}$, and ϵ_0 is the energy at the Fermi boundary (see, for example, [9]). Hence, for $T \sim 3^\circ$ we have $l_e \sim 3$ cm, while in a superconductor l_e is even larger, due to the appearance of the factor $\exp(2\Delta/kT)$ (2Δ is the width of the gap formation of two single-particle excitations). For $u_2 \sim 10^4$ (see below) and $l_e \sim 10$ we arrive at the wholly unrealistic conditions $\omega \ll l_e/2\pi u_2 \sim 10^4$ and $l_s \gg 10^8/\omega \sim 10^4 - 10^5$ cm.

Nevertheless, the possibility of the occurrence of second sound in superconductors merits more careful attention. The fact is, that the spectrum and character of the excitations in the superconducting state differ radically from those prevailing in the normal state. Thus, in the isotropic model [8] the single-particle excitations in superconductors possess the following energy and velocity:

$$E = \sqrt{\epsilon^2 + \Delta^2}, \quad \epsilon = \frac{p^2 - p_0^2}{2m} = \frac{p^2}{2m} - \epsilon_0;$$

$$v = \frac{dE}{dp} \approx \frac{v_0 \epsilon}{E}, \quad v_0 = \frac{p_0}{m}.$$

Since in the simplest case the number of collisions is proportional to v , then in the superconducting state $\nu^{(s)} = \nu^{(n)} v/v_0$. If the Fermi surfaces are complex, one can assume that for a sufficiently large number of excitations $\nu^{(s)}$ is much smaller than $\nu^{(n)}$. It is unlikely, however, that ν should change in these circumstances by many orders of magnitude.

The situation may prove to be more favorable with regard to transverse collective excitations (excitons), which appear under certain conditions in superconductors* (see [8,9,10-13]). The scattering

of excitons by impurities is diminished for a number of reasons. Thus, in the long-wave region collective excitations are in a number of cases only weakly scattered, since the energy of their interaction with impurities contains the wave vector as a factor. As an example, we may point to the scattering of phonons and long-wave excitons (photons) in non-conductors. As regards excitons in superconductors in the long-wave region, neither the character of the excitations themselves* nor their law of scattering is as yet sufficiently clear. This region, however, is of least importance, since the corresponding excitons are extremely few in number and cannot contribute significantly to ρ_n . However, for excitons having a sufficiently large wave vector q , these playing the principal role, the effective cross section for scattering by impurities can fall sharply, as a consequence of the large size of the excitons.

In fact, to a certain approximation [12,13] an exciton can be regarded as an excited Cooper "pair" having the wave function $\Psi = e^{iq \cdot R} \varphi(r)$, where $R = (m_1 r_1 + m_2 r_2)/(m_1 + m_2)$ locates the center of mass of the pair and $r = r_2 - r_1$ is the relative separation of its components. We shall now consider the scattering of the pairs by impurities, in which the interaction energy is

$$V = A_1 D_1(r_1) + A_2 D_2(r_2), \quad \int D_{1,2}(r) dr = 1.$$

Then, in the Born approximation, the transport cross section for elastic scattering of the pairs is ($M = m_1 + m_2$, $d\Omega = \sin \theta d\theta d\varphi$)

$$\sigma_{tr} = \frac{M^2}{4\pi^2 \hbar^4} \int (1 - \cos \theta) \left| \int \left\{ A_1 D_1 \left(R - \frac{m_2}{M} r \right) + A_2 D_2 \left(R + \frac{m_1}{M} r \right) \right\} e^{i(q-q') \cdot R} |\varphi(r)|^2 dR dr \right|^2 d\Omega. \quad (4)$$

The size of the pairs is characterized by the parameter a , which is of the order of or greater than $\xi_0 \sim 10^{-4} - 10^{-5}$ cm. $D_{1,2}$ can therefore be replaced by a δ -function. Setting, for simplicity, $\varphi(r) = \varphi(r)$, $m_1 = m_2 = M/2$, and $A_1 = A_2 = A$ we have

$$\sigma_{tr} = \frac{32\pi M^2 A^2}{\hbar^4} \int_0^\pi \int_0^\infty \frac{\sin(qr \sin(\theta/2)) |\varphi(r)|^2}{q \sin(\theta/2)} r dr \times (1 - \cos \theta) \sin \theta d\theta. \quad (5)$$

If $qa \ll 1$, then $\sigma_{tr} = 4M^2 A^2 / \pi \hbar^4$, since

$$4\pi \int |\varphi(r)|^2 r^2 dr = 1.$$

*The conclusion that the excitation spectrum of a superconductor is a mixed one, and incorporates both Fermi and Bose (photon) excitations was discussed some time ago. [6] Only the microscopic theory, however, was under consideration (the term "photon" used in [6] for electric waves in the medium is identical in present-day terminology with the term "exciton;" [14] in [6] retardation was regarded as of fundamental importance, while in [8,10-13] the excitons correspond to poles of the permittivity $\epsilon(\omega, k)$.

*For non-conductors, it is known (see, for example, [14]) that in the long-wave region one must take retardation into account. We are not clear as to the region of applicability of the results of [12,13], especially for long waves.

For $qa \gg 1$, on the other hand, the cross section σ_{tr} is, generally speaking, extremely small. At the same time, for excitons whose energy E_e is near 2Δ , the size of the pairs increases greatly, and $a \gg \xi_0$ (see [12]). To this must be added the fact that the velocity dE_e/dq of the pairs is also extremely small when q is large and $E \sim 2\Delta$ (see the dispersion equation (3.22) in [13]). There is thus an established basis for assuming that a significant fraction of the pairs is extremely weakly scattered by impurities; and the hypothesis that for excitations of the exciton type $\nu^{(s)}$ is smaller by many orders of magnitude than $\nu^{(n)}$ does not appear improbable, although it can by no means be considered as proven.

In connection with what has been said, let us suppose that there can exist a group of excitations, most probably excitons, possessing low friction. These excitations, then, could make a considerable contribution to the transport processes, and make possible as well the propagation of second sound in superconductors (the latter, of course, being less likely). If we assume that the exciton mean free path satisfies certain conditions, which are evident from what has already been said at the beginning of this article, then the second sound velocity $u_{2e} = \sqrt{\rho_s S_e^2 T / \rho \rho_{ne} C_e}$, where S_e , C_e and ρ_{ne} are the contributions these excitations make to S , C , and ρ_n (as before, however, ρ_s and ρ refer to all of the conduction electrons, and, of course, $\rho \neq \rho_s + \rho_{ne}$).

As we pass from one metal to another ρ_{ne} , S_e and C_e will obviously change greatly; only for Pb and Hg among the metals studied are these quantities relatively large. Anomalous behaviour of the specific heat is, in fact, observed in Pb and Hg, [15] and microwave absorption sets in for $\hbar\omega < 2\Delta$ (see [16,17]). These two effects, if they are to be associated with exciton excitations, cannot be explained within the framework of the theory of weak binding of the electrons with the lattice. [8,12] In both Pb and Hg, on the other hand, the binding is relatively strong, and the usual approximation is probably not applicable (we note that for Pb, Hg and Sn, respectively, the ratio $10^3 T_c / \theta_D$ has the values 76, 52, and 19 (see [17]); here T_c and θ_D are the critical and Debye temperatures respectively). From the point of view of this hypothesis, second sound propagation may most probably occur in Pb and Hg. If all of the values for Pb are taken for the simplest model of a superconductor*, then, from Eq. (3), $u_2 = 2.2 \times 10^4$ at

*The following values have been used: [8] $\rho = mn = 1.8 \times 10^{-5}$ ($\lambda_L(0) = 3.7 \times 10^{-6}$), $\gamma T_c = C_0 = 1.22 \times 10^4$, $T/T_c = 0.54$, $S/C_0 = 0.17$, $C/C_0 = 0.59$, $\rho_n/\rho = 1 - \rho_s/\rho = 0.21$. For

$T = 3.9^\circ$. It may be presumed that u_{2e} does not differ from u_2 by more than an order of magnitude, since the quantity $S_e^2 / \rho_{ne} C_e$ is considerably less sensitive to changes in the various parameters than ρ_{ne} , S_e , or C_e themselves.

If a group of excitations having sufficiently small friction exists at all, then propagation of second sound is possible in principle, even for extremely small values of ρ_{ne} and S_e , although in the latter case this sound would be more difficult to excite and detect. As regards Pb and Hg, one may expect an appreciable contribution by the exciton excitations to the thermal conductivity, as well as to the specific heat, resulting from convective heat transfer. The possibility of such an effect for Pb was discussed some time ago, [18,19,2,3] but seemed highly unlikely on theoretical grounds. On the other hand, an alternative explanation of the experiments with Pb — Bi alloys, based upon consideration of the part played by the thermal conductivity of the lattice, meets with difficulties [3] (in order to explain the data of [19], the lattice thermal conductivity for an alloy containing 0.5% Bi, to take one example, would have to be higher than for an alloy with 0.02% Bi).

In the light of the ideas advanced above, it seems probable to us that convective heat transport has actually been observed in Pb. We note that in non-cubic metals (here, mercury is of especial interest) under certain conditions a total current j must flow during the heat transfer process, and, consequently, that magnetic measurements are also possible. [1] During convective heat transport the normal fluid is transformed at the boundary of the superconductor into superfluid; as in the case of He II, therefore, a temperature discontinuity can arise.

We should also point out another possible role of the convective mechanism, viewed as an explanation of the peculiar behavior of the thermal conductivity in the intermediate state. [3,18,19] With the density ρ_n there is associated a charge $(e/m)\rho_n$; on the other hand, the mean free path of the excitons may be large. Their contribution, therefore, to the "normal conductivity" and to the surface impedance Z might be significant even for small ρ_{ne} . In this view, attention is drawn to the fact that in a number of cases the quantity $R = \text{Re}Z$ exceeds the value calculated without taking exciton excitations into account, [20] the complicated dependence of Z upon an external magnetic field [21]

Sn: $C_0 = 4.1 \times 10^3$, $\rho = 2.05 \times 10^{-5}$ ($\lambda_L(0) = 3.55 \times 10^{-6}$), and $u_2 = 8.6 \times 10^3$ cm/sec (for $T/T_c = 0.54$, $T = 2^\circ$).

also remains unclear. (In computing the effect of a field upon the current in a superconductor, Miller^[22] made use of an assumption which was tantamount to neglecting transverse excitons).

Finally, let us turn to the problem of the Knight frequency shift in superconductors. Experimental data^[23,24] testify to the fact that for $T \ll T_c$ the Knight shift is smaller by only 20–30% than in the normal state. At the same time, it follows from theory that at $T = 0$ this shift must be equal to zero.^[8,25,26] In view of this contradiction, Abrikosov and Gor'kov^[26] not only criticized certain hypotheses concerning the origins of the Knight shift in superconductors, but also expressed general doubts concerning the reality of the effect itself. It seems to us, however, that as regards the experiments^[24] discussed in^[26] these objections do not apply at all. In these experiments the samples (fine particles of Sn) were of graded dimensions, much smaller (by tens of times) than the field penetration depths. Furthermore, while the fields employed exceeded 1200 oersteds, they were nevertheless considerably below the critical field [$H_c(0) \approx 2.5 \times 10^4$]. There is no basis, therefore, for assuming that superconductivity was destroyed in parts of the samples. Despite the small sizes of the particles, their critical temperature was 3.71° , which is lower by only 0.02° than the T_c of bulk Sn. It must therefore be presumed that the fundamental state of the superconductor in the particles investigated in the absence of a field was the same as in the bulk metal, and conformed to the representation of the contemporary theory of superconductivity, which for Sn agrees well with experiment.^[8]

An attempt can be made to explain the presence of a Knight shift as due to the effect of the strong magnetic field used in these experiments^[23,24] upon the exciton levels. From the constancy of T_c noted above it may be presumed that for small samples these levels are close to the levels in the bulk superconductor (the smallest particle size in reference 24 is ~ 40 Å, which is still considerably greater than the electron wavelength at the Fermi boundary $\hbar/p_0 \sim 1$ Å). In the particles, however, it is inappropriate to speak of the motion of the exciton center of mass, and the energy of interaction of the exciton with the field H is

$$-\frac{e}{2m_{\text{eff}}c}HL + \frac{e^2}{16m_{\text{eff}}c^2}([Hr])^2. \quad *$$

Here, L is the angular momentum of the exciton, which, it is assumed, is composed of two

identical quasi-particles of charge e and mass m_{eff} . Due to the small size of the samples the diamagnetic effect is inconsequential; thus, even for a particle of dimensions $\sim 3 \times 10^{-6}$ cm

$$\Delta E \sim (e^2 / 16mc^2) H^2 \bar{r}^2 \sim 10^{-24} H^2.$$

At the same time, the depression, associated with paramagnetism, of the lower of the Zeeman sublevels of the exciton level is $\Delta E = -e\hbar l H / 2m_{\text{eff}}c$, where $\hbar l$ is the angular momentum of the exciton. Even for $l = 1$ and $m_{\text{eff}} = m$ we have $\Delta E \approx 10^{-20} H > 10^{-17}$, for $H > 10^3$. If the lower exciton level is sufficiently low ($E_e \ll 2\Delta \sim 10^{-15}$), $m_{\text{eff}} \ll m$, or excitons are present with $l \gg 1$, then in the fields used ($H > 10^3$) the lower Zeeman sublevel may reach the energy of the ground state. The latter thereby becomes unstable, and another state should arise which may show paramagnetic susceptibility.

The suggestion just advanced seems to us scarcely probable, but it deserves attention in view of the fact that the experiments of^[23,24] have not been explained. Moreover, the hypothesis concerning the role of Zeeman splitting of the exciton levels can be checked; from this viewpoint superconductors should show no Knight shift in a sufficiently weak field.

The problem of excitons in superconductors is among those which have been least studied, while for the most interesting case of metals of the type of Pb and Hg no theory exists at all. For this reason, therefore, it seemed appropriate to us to devote the present article to questions of merely hypothetical character. Moreover, even if our suggestions turn out to correspond only in part to reality, then the part played by excitons in superconductors may be considered to be as great as in the case of semiconductors. Here, evidently, a whole province is laid open for new experimental and theoretical investigations.

In conclusion, I take this opportunity to thank G. P. Motulevich for discussing the question she has raised concerning the possibility of observing second sound in superconductors, and L. P. Gor'kov for his comments.

¹ V. L. Ginzburg, JETP 14, 177 (1944), J. Phys. U.S.S.R. 8, 148 (1944); Sverkhprovodimost' (Superconductivity), AN SSSR, 1946.

² V. L. Ginzburg, JETP 21, 979 (1951).

³ P. G. Klemens, Handbuch der Physik 14, 198 (1956); Proc. Phys. Soc. A66, 576 (1953).

⁴ B. T. Geilikman, JETP 34, 1042 (1958), Sov. Phys. JETP 34, 721 (1958).

* $HL = H \cdot L$; $[Hr] = H \times r$.

⁵ L. D. Landau and E. M. Lifshitz, *Mekhanika sploshnykh sred* (Mechanics of Continuous Media), Gostekhizdat 1953, ch. 16.

⁶ V. L. Ginzburg, *Usp. Fiz. Nauk* **48**, 25 (1952), *Fortschr. Phys.* **1**, 101 (1953).

⁷ C. J. Gorter, *Progr. in Low Temperature Physics* (Amsterdam) **1**, 1 (1955).

⁸ J. Bardeen and J. R. Schrieffer, *Progr. in Low Temperature Physics*, (Amsterdam) **3**, 170 (1961).

⁹ V. L. Ginzburg and V. P. Silin, *JETP* **29**, 64 (1955), *Sov. Phys. JETP* **2**, 46 (1956).

¹⁰ Bogolyubov, Tolmachev, and Shirkov, *Novyĭ metod v teorii sverkhprovodimosti* (New Method in Superconductivity Theory), AN SSSR, 1958.

¹¹ P. W. Anderson, *Phys. Rev.* **112**, 1900 (1958).

¹² T. Tsuneto, *Phys. Rev.* **118**, 1029 (1960).

¹³ A. Bardasis and J. R. Schrieffer, *Phys. Rev.* **121**, 1050 (1961).

¹⁴ Ginzburg, Rukhadze, and Silin, *FTT* **3**, 1835 (1961), *Sov. Phys. Solid State* **3**, 1337 (1961).

¹⁵ Decker, Mapother, and Shaw, *Phys. Rev.* **112**, 1888 (1958).

¹⁶ D. M. Ginsberg and M. Tinkham, *Phys. Rev.* **118**, 990 (1960).

¹⁷ P. L. Richards and M. Tinkham, *Phys. Rev.* **119**, 575 (1960).

¹⁸ K. Mendelssohn, *Progr. in Low Temperature Physics* (Amsterdam) **1**, 184 (1955).

¹⁹ J. L. Olsen, *Proc. Phys. Soc.* **A65**, 518 (1952).

²⁰ Abrikosov, Gor'kov, and Khalatnikov, *JETP* **37**, 187 (1959), *Sov. Phys. JETP* **10**, 132 (1960).

²¹ M. Spiewak, *Phys. Rev.* **113**, 1479 (1959).

²² P. B. Miller, *Phys. Rev.* **121**, 435 (1961).

²³ F. Reif, *Phys. Rev.* **106**, 208 (1957).

²⁴ G. M. Androes and W. D. Knight, *Phys. Rev.* **121**, 779 (1961).

²⁵ K. Yosida, *Phys. Rev.* **110**, 769 (1958).

²⁶ A. A. Abrikosov and L. P. Gor'kov, *JETP* **39**, 480 (1960), *Sov. Phys. JETP* **12**, 337 (1961).

Translated by S. D. Elliott

τ DECAY AND $\pi\pi$ COUPLING

J. WOLF and W. ZOELLNER

Joint Institute for Nuclear Research

Submitted to JETP editor March 31, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 835-841 (September, 1961)

Integral equations for the probability of τ decay are derived. An effective range analysis gives s-wave scattering lengths for the τ interaction; from which the experimental τ -decay spectrum can be synthesized. Satisfactory agreement with experiment is obtained for $a_2 \approx 0.2$ and $a_0 \approx 0.3$. In conjunction with the integral equation for $\pi\pi$ scattering, these values speak in favor of the existence of a ($T = 2$) resonance in $\pi\pi$ interaction.

1. INTRODUCTION

THE τ decay is one of the few processes which enable us to investigate the $\pi\pi$ interaction without the influence of other particles. The small kinetic energy of the emitted pions facilitates the theoretical analysis, and the rather extensive experimental data enable us to compare theory with experiment.

To investigate the τ decay we propose to use the Mandelstam representation. It must be noted, however, that such a representation is in general valid for decay processes only in a certain approximation.^[1] Fubini and Stroppolini have shown^[2] that the imaginary part of the spectral functions corresponds to processes with three particles in intermediate states.* The imaginary part makes no contribution in the two-particle approximation, which we use. In the present paper we consider the consequences of such an approximation. Analogous considerations were used by Khuri and Treiman.^[3]

For the determination of the τ decay we shall henceforth write out integral equations with account of the interaction between pions in p states. It thus becomes possible in principle to verify the solutions of the integral equations of $\pi\pi$ scattering.

To estimate such $\pi\pi$ -interaction data as the scattering lengths, we can neglect the p waves. A comparison of the calculated matrix element with the experimental spectrum of the τ decay yields an estimate for the scattering lengths of the s waves of the $\pi\pi$ interaction. Using these scattering lengths and the integral equations from^[4] we can obtain further information on the $\pi\pi$ interaction.

*The authors are grateful to G. Bonneva for a preprint in which this problem is also discussed.

2. INTEGRAL EQUATIONS WITH ACCOUNT OF p WAVES

To obtain equations for the amplitudes of the τ^+ and τ'^+ decay into three pions we consider the following reactions:

$$\tau^+ + \pi_1^- \rightarrow \pi_2^+ + \pi_3^-, \quad (\text{I})$$

$$\tau^+ + \pi_2^- \rightarrow \pi_3^- + \pi_1^+, \quad (\text{II})$$

$$\tau^+ + \pi_3^- \rightarrow \pi_1^+ + \pi_2^+, \quad (\text{III})$$

$$\tau'^+ + \pi_1^0 \rightarrow \pi_2^0 + \pi_3^+, \quad (\text{I}')$$

$$\tau'^+ + \pi_2^0 \rightarrow \pi_3^+ + \pi_1^0, \quad (\text{II}')$$

$$\tau'^+ + \pi_3^0 \rightarrow \pi_1^0 + \pi_2^0. \quad (\text{III}')$$

The invariant variables of these processes assume in c.m.s. of reaction (III) the form

$$s_1 = \gamma - 2q_3^2 + 2p_3q_3z_3, \quad s_2 = \gamma - 2q_3^2 - 2p_3q_3z_3, \\ s_3 = 4(q_3^2 + \mu^2). \quad (1)$$

Here m and μ denote respectively the K-meson and pion masses; $\gamma = (m^2 - \mu^2)/2$; p_3 and q_3 are the momenta of the particles before and after collision; z_3 denotes the cosine of the scattering angle. Analogous relations are true also in the c.m.s. of the other reactions.

The variables s_i satisfy the condition

$$s_1 + s_2 + s_3 = m^2 + 3\mu^2. \quad (2)$$

We shall use also the following invariant combinations:

$$2\eta_1 = s_2 - s_3 = 4p_1q_1z_1, \\ 2\eta_2 = s_3 - s_1 = 4p_2q_2z_2, \\ 2\eta_3 = s_1 - s_2 = 4p_3q_3z_3. \quad (3)$$

We need the values of the invariant variables for $z_1 = \pm 1$. In this case we obtain

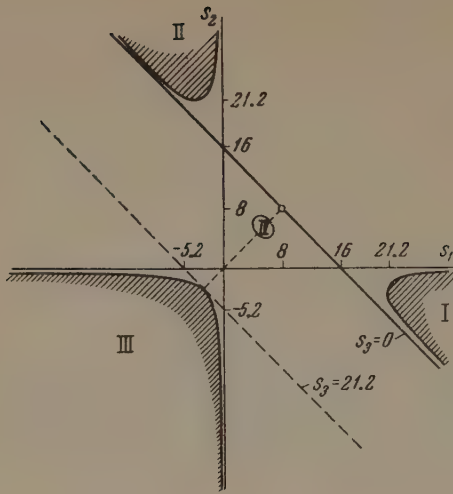


FIG. 1

$$s_1 s_2 s_3 = \mu^2 (m^2 - \mu^2)^2. \quad (4)$$

Figure 1 is a plot of Eq. (4). Here I, II, and III are the physical regions of the corresponding scattering processes, while region IV corresponds to the decay process. In the case of an unreal K-meson mass ($m = 3\mu$) this region contracts to a point. For real $s_1 = 4 (\nu_1 = 1)^*$ with $m = 3.6\mu$ the product $(p_i q_i)$ becomes complex when

$$-1 < \nu_i < 0, \quad 0.7 < \nu_i < 4.3.$$

Let us assume the following representation

$$A(s_1, s_2, s_3) = \frac{1}{\pi} \int_4^\infty \frac{\rho(s') + \eta_1 \sigma(s')}{s' - s_1} ds' + \frac{1}{\pi} \int_4^\infty \frac{\rho(s') - \eta_2 \sigma(s')}{s' - s_2} ds' + \frac{1}{\pi} \int_4^\infty \frac{\lambda(s')}{s' - s_3} ds', \quad (5)$$

which for $m \leq 3$ can be obtained, as the Cini-Fubini approximation^[5] for p waves, from the ordinary Mandelstam representation. For $\sigma = 0$ in the s-wave approximation, relation (5) was used in several investigations.^[3,6]

From (5) and the unitarity conditions for processes I and III we obtain equations that define the functions ρ , λ , and σ . We write the partial-wave expansion of the scattering amplitudes in the form

$$A(\nu, z) \approx A_0(\nu) + 3pqz A_1(\nu) \quad (6)$$

and define the s and p waves respectively as

$$A_0(\nu) = \frac{1}{2} \{A(+) + A(-)\}, \quad A_1(\nu) = \frac{1}{6pq} \{A(+) - A(-)\}. \quad (7)$$

Here $A(\pm) = A(\nu, z = \pm 1)$; we neglect the d waves and higher waves. We denote the amplitudes

of the s waves of processes III and I by F_0 and G_0 , and the amplitudes of processes III' and I' by f_0 and g_0 . The amplitudes of the p waves are denoted respectively by G_1 and g_1 .

From (5), (7), and (1) we obtain

$$F_0(s) = \Lambda^F + \frac{s-4}{\pi} \int_4^\infty \frac{ds' \lambda(s')}{(s'-4)(s'-s)} + \frac{2}{\pi} \int_4^\infty \frac{ds' \rho(s')}{s'-\gamma} \frac{(\alpha-\gamma)(s'-\alpha) + 4p^2 q^2}{(s'-\alpha)^2 - 4p^2 q^2} - \frac{1}{\pi} \int_4^\infty ds' \sigma(s') \frac{(6\nu+4-\gamma)(s'-\alpha) + 4p^2 q^2}{(s'-\alpha)^2 - 4p^2 q^2} - \frac{\gamma-4}{\pi} \int_4^\infty \frac{ds' \sigma(s')}{s'-\gamma}, \quad (8)$$

$$G_0(s) = \Lambda^G + \frac{s-4}{\pi} \int_4^\infty \frac{ds' \rho(s')}{(s'-4)(s'-s)} + \frac{1}{\pi} \int_4^\infty \frac{ds'}{s'-\gamma} \{\rho(s') + \lambda(s')\} \frac{(\alpha-\gamma)(s'-\alpha) + 4p^2 q^2}{(s'-\alpha)^2 - 4p^2 q^2} + \frac{1}{2\pi} \int_4^\infty ds' \sigma(s') \frac{(6\nu+4-\gamma)(s'-\alpha) + 4p^2 q^2}{(s'-\alpha)^2 - 4p^2 q^2} + \frac{\gamma-4}{2\pi} \int_4^\infty \frac{ds' \sigma(s')}{s'-\gamma}, \quad (9)$$

where $\alpha = (m^2 - 1)/2 - 2\nu = \gamma - 2\nu$. For the amplitude F_0 the subtraction is carried out at the point $s_1 = s_2 = \gamma$, $s_3 = 4$, for G_0 at the point $s_1 = 4$, $s_2 = s_3 = \gamma$ where $\Lambda^F = A(\gamma, \gamma, 4)$ and $\Lambda^G = A(4, \gamma, \gamma)$. These complex constants are related by Eq. (5). We note that (8) and (9) actually contain only the real parts of Λ^F and Λ^G , and therefore only one constant appears. For p waves we obtain an analogous equation.

We note that (8) and (9) contain only even powers of pq , and therefore the imaginary part vanishes in the regions where $(pq)^2$ becomes negative. For ρ , λ , and σ the unitarity conditions yield

$$\rho = \sqrt{\frac{\nu}{\nu+1}} \left\{ G_0 \Pi_0^{0*} - \frac{1}{6} F_0 (\Pi_0^{0*} - \Pi_0^{2*}) \right\}, \quad (10)$$

$$\lambda = \sqrt{\frac{\nu}{\nu+1}} F_0 \Pi_0^{2*}, \quad \frac{2}{3} \sigma = \sqrt{\frac{\nu}{\nu+1}} G_1 \Pi_1^{1*}.$$

$\Pi_l^T(\nu)$ are the partial amplitudes of the $\pi\pi$ scattering for the isotopic spin T . In the derivation of (10) we use the selection rule $|\Delta T| = 1/2$, which yields the following relations between the amplitudes that determine the τ and τ' decays:

$$F_0 = 2g_0, \quad G_0 = f_0 + g_0, \quad G_1 = -g_1. \quad (11)$$

Equations (8) and (9) can be written in the form of dispersion relations in the ν plane. It is then

*We put $\nu_1 = q_1^2$ and $\mu^2 = 1$.

easy to verify that the crossing amplitudes in the range of the arguments $1 < \nu^C < 4.3$ [where $\nu^C = \alpha(\nu) - 2pq$] make no contribution to the dispersion relations. Therefore the possible resonance of the $\pi\pi$ interaction in the p wave, expected approximately in this region, has practically no effect on the values of F_0 and G_0 . It must be remembered that the region of the physical τ decay is limited to $\nu \approx 0.7$, and therefore amplitudes with large values of ν will exert only an insignificant influence on the decay. However, in spite of the weak influence of the p waves, their account may be important for a verification of solutions of the integral equations of the $\pi\pi$ interaction.

3. EQUATIONS FOR s AMPLITUDES AND ANALYSIS IN THE EFFECTIVE-RADIUS APPROXIMATION

Neglecting the terms containing the p wave in (8) and (9), we can obtain a system of integral equations for the s waves only. However, to obtain a numerical estimate it is more convenient to use a system of integral equations obtained from (5) by integrating along the lines $z_i = 0$. We therefore choose instead of (6) and (7) the following definition for the s waves:

$$A(\nu, z = 0) \approx A_0(\nu). \quad (12)$$

This leads to the relations

$$\begin{aligned} \operatorname{Re} F_0(\nu) &= \Lambda + P \frac{1}{\pi} \int_0^\infty d\nu' \lambda(\nu') \left\{ \frac{1}{\nu' - \nu} - \frac{1}{\nu' - \nu_0} \right\} \\ &+ P \frac{2}{\pi} \int_0^\infty d\nu' \rho(\nu') \left\{ \frac{1}{\nu' - (1-\nu)/2} - \frac{1}{\nu' - \nu_0} \right\}, \\ \operatorname{Re} G_0(\nu) &= \Lambda + P \frac{1}{\pi} \int_0^\infty d\nu' \rho(\nu') \left\{ \frac{1}{\nu' - \nu} - \frac{1}{\nu' - \nu_0} \right\} \\ &+ P \frac{1}{\pi} \int_0^\infty d\nu' \{ \rho(\nu') + \lambda(\nu') \} \left\{ \frac{1}{\nu' - (1-\nu)/2} - \frac{1}{\nu' - \nu_0} \right\}. \end{aligned} \quad (13)$$

Subtraction is made at the point $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 1/3$, where $\Lambda = \operatorname{Re} A(\nu_0, \nu_0, \nu_0)$. In the region of physical τ decay, we have ($0 \leq \nu \leq 1$)

$$\operatorname{Im} F_0(\nu) = \lambda(\nu) + 2\rho(1 - \nu/2), \quad (15)$$

$$\operatorname{Im} G_0(\nu) = \rho(\nu) + \rho(1 - \nu/2) + \lambda(1 - \nu/2). \quad (16)$$

Relations (13) and (16) are used for analysis in the effective-radius approximation. This approximation can be quite good in view of the weak energy dependence of the τ decay probability. In this way Khuri and Treiman^[3] obtained by comparison with experiment estimates for the difference in the

s-scattering lengths in the $\pi\pi$ interaction. Their research, however, must be made more accurate, for example, by taking into account the influence of the imaginary part on the decay probability.

We start from the following approximation:

$$\sqrt{\frac{\nu}{\nu+1}} \Pi_0^T(\nu) \approx \frac{a_T \sqrt{\nu}}{1 - ia_T \sqrt{\nu}} \approx a_T \sqrt{\nu}, \quad F_0 \approx G_0 \approx \Lambda, \quad (17)$$

where a_T is the scattering length.

Substituting (17) in (15) and (17), we obtain

$$\frac{1}{\Lambda} \operatorname{Re} F_0(\nu) = 1 + \frac{a_2}{\pi} Z_1(\nu) + \frac{5a_0 + a_2}{3\pi} Z_2(\nu), \quad (18)$$

$$\frac{1}{\Lambda} \operatorname{Re} G_0(\nu) = 1 + \frac{5a_0 + a_2}{6\pi} Z_1(\nu) + \frac{5a_0 + 7a_2}{6\pi} Z_2(\nu);$$

$$\frac{1}{\Lambda} \operatorname{Im} F_0(\nu) = a_2 \sqrt{\nu} + \frac{5a_0 + a_2}{3} \sqrt{\frac{1-\nu}{2}} \quad (0 \leq \nu \leq 1), \quad (19)$$

$$\frac{1}{\Lambda} \operatorname{Im} G_0(\nu) = \frac{5a_0 + a_2}{6} \sqrt{\nu} + \frac{5a_0 + 7a_2}{6} \sqrt{\frac{1-\nu}{2}} \quad (0 \leq \nu \leq 1),$$

where

$$\begin{aligned} Z_1(\nu) &= \sqrt{\nu} \ln \frac{1 - \sqrt{\nu}}{1 + \sqrt{\nu}} - \sqrt{\nu_0} \ln \frac{1 - \sqrt{\nu_0}}{1 + \sqrt{\nu_0}}, \\ Z_2(\nu) &= \sqrt{\frac{1-\nu}{2}} \ln \frac{1 - \sqrt{(1-\nu)/2}}{1 + \sqrt{(1-\nu)/2}} - \sqrt{\nu_0} \ln \frac{1 - \sqrt{\nu_0}}{1 + \sqrt{\nu_0}}. \end{aligned} \quad (20)$$

Using (18) and (19) as well as the approximation

$$\sqrt{\frac{\nu}{\nu+1}} \Pi_0^T(\nu) \approx \sqrt{\nu} a_T + i\nu a_T^2 \quad (17')$$

for the amplitudes of the $\pi\pi$ interaction, we obtain expressions for ρ and λ , which we substitute in (5). After integration and expansion in powers of ν_i (with account of the fact that $\nu_1 + \nu_2 + \nu_3 = 1$) the real part of the decay amplitude assumes the form

$$\frac{\pi}{\Lambda} \operatorname{Re} A = \operatorname{const} - \frac{\pi}{\alpha} R(a_2, a_0) \nu_3. \quad (21)$$

Here

$$\begin{aligned} \frac{\pi}{\alpha} R(a_2, a_0) &= \frac{5}{3} (a_2 - a_0) + \frac{5}{3} (a_2^2 - a_0^2) \\ &- \frac{5}{9\pi} (a_2^2 + a_2 a_0 - 2a_0^2) - \frac{5}{3} (a_2^3 - a_0^3) \end{aligned} \quad (22)$$

and $\alpha = 1/4 (m^2 - 1)^2 - 1 = 0.7$ when $m^2 \approx 13$.

For ν_i we have

$$\nu_i = \alpha (1 - t_i); \quad (23)$$

here t_i is the kinetic energy of the π meson with index i in the rest system of the τ meson.

Carrying out the subtraction at the symmetrical point $t_i = t_0 = 1/2$, where $\Lambda = \operatorname{Re} A(t_0)$, we obtain

$$\Lambda^{-2} \{ \operatorname{Re} A(t_3) \}^2 = 1 + (2t_3 - 1) R(a_2, a_0) + \Gamma^2. \quad (24)$$

Here

a_2	-1	-0.3	-0.333	-0.2	0	0	0	0.1	0.1	0.15
a_0	-0.465	-1	-0.135	-0.27	-0.5	-0.3	0	-0.65	-0.45	-0.4
σ ($\nu_3 = 0; \nu_1$ $= \nu_2 = 0.5$)	0.64	0.45	0.89	1.21	1.23	1.08	1	1.41	1.28	1.28
σ ($\nu_3 = 0.7; \nu_1$ $= \nu_2 = 0.15$)	1.41	0.01	1.05	0.68	0.79	0.85	1	0.71	0.79	0.81
a_2	0.2	0.2	0.25	0.3	0.333	0.333	0.4	0.5	0.6	0.7
a_0	-0.3	0.2	-0.35	-0.5	-0.2	0.135	-0.3	0	-0.8	0.11
σ ($\nu_3 = 0; \nu_1$ $= \nu_2 = 0.5$)	1.22	1.20	1.32	1.13	1.23	1.0	1.36	1.11	1.45	0.95
σ ($\nu_3 = 0.7; \nu_1$ $= \nu_2 = 0.15$)	0.82	0.87	0.78	0.76	0.83	0.78	0.79	0.95	0.49	1.03

$$\Gamma^2 = \frac{1}{4} (2t - 1)^2 \{R(a_2, a_0)\}^2. \quad (25)$$

For the imaginary part of the decay amplitude with

$$\text{Im } A(\nu_1, \nu_2, \nu_3) = \rho(\nu_1) + \rho(\nu_2) + \lambda(\nu_3) \quad (26)$$

we obtain from (11), (17'), (18), and (19) the expression

$$\begin{aligned} \frac{1}{\Lambda} \text{Im } A(\nu_1, \nu_2, \nu_3) \equiv I &= (\sqrt{\nu_1} + \sqrt{\nu_2}) \left\{ \frac{1}{6} (5a_0 + a_2) \right. \\ &\quad \left. - \frac{5}{54\pi} (a_2 - a_0) (4a_0 - a_2) \right\} + \sqrt{\nu_3} a_2 \left\{ 1 + \frac{5}{9\pi} (a_2 - a_0) \right\} \\ &\quad + (\nu_1^{1/2} + \nu_2^{1/2}) \left\{ \frac{4}{6} (5a_0^2 + a_2^2) + \frac{5}{18\pi} (a_2 - a_0) (4a_0 - a_2) \right\} \\ &\quad + \nu_3^{1/2} a_2 \left\{ a_2^2 - \frac{5}{3\pi} (a_2 - a_0) \right\} \\ &\quad + \left(\nu_1 \sqrt{\frac{\nu_2 + \nu_3}{2}} + \nu_2 \sqrt{\frac{\nu_1 + \nu_3}{2}} \right) \\ &\quad \times \left\{ \frac{5}{9} a_0^2 (a_0 + 2a_2) + \frac{1}{18} a_2^2 (5a_0 + a_2) \right\} \\ &\quad + \nu_3 \sqrt{\frac{\nu_1 + \nu_2}{2}} \left\{ \frac{4}{3} a_2^2 (5a_0 + a_2) \right\}. \end{aligned} \quad (27)$$

With the aid of (23) and (24) we obtain for the square of the Feynman amplitude*

$$\begin{aligned} \sigma' &\equiv \Lambda^{-2} \{(\text{Re } A)^2 + (\text{Im } A)^2\} \\ &= 1 + (2t_3 - 1) R(a_2, a_0) + K(a_2, a_0; \nu_i), \end{aligned} \quad (28)$$

where

$$K(a_2, a_0; \nu_i) = \Gamma^2 + I^2. \quad (29)$$

We shall compare this expression with the quantity σ , which adequately describes the experimental data^[7]:

$$\sigma \equiv |M|^2 = 1 + 0.2(2t_3 - 1). \quad (30)$$

The quantities σ and σ' are related:

$$\sigma = \sigma' / [1 + I^2(\nu_0)]. \quad (31)$$

We list the data on the dependence of the theoretical value of σ on the scattering lengths a_2, a_0 [by definition $\sigma(\nu_1 = \nu_2 = \nu_3 = 1/3) = 1$].

*The authors have been informed of Gribov's paper^[9]. We note that the diagrams which he takes into account arise in our case by iteration of the resultant system of equations.

Agreement with the experimental energy dependence can be obtained, for example, when $a_2 \approx 0.2$ and $a_0 \approx -0.3$. In this case $R(0.2; -0.3) = 0.16$ and

$$\begin{aligned} \sigma(\nu_3 = 0, \nu_1 = \nu_2 = 0.5) &\approx 1.2, \\ \sigma(\nu_3 = \nu_1 = \nu_2 = 1/3) &= 1, \\ \sigma(\nu_3 = 0.7, \nu_1 = \nu_2 = 0.15) &\approx 0.8, \end{aligned} \quad \frac{1}{\Lambda^2} (\text{Im } A)^2 = \begin{cases} 0.07, \\ 0.02, \\ 0.00. \end{cases} \quad (32)$$

The experimental relation (30) for the energies used in (32) yields indeed $\sigma = 1.2, 1$, and 0.8 .

Figure 2 is a plot of the equations $R(a_2, a_0) = 0.2$ and $I(\nu_i = 1/3) = 0$. If we allow a 40 percent error in the determination of the energy-dependent part of σ , we find that the values of the area shaded in Fig. 2 duplicate the experimental data with a sufficient degree of accuracy.

We have given values for σ at equal energies $\nu_1 = \nu_2$. We note that when $\nu_1 \neq \nu_2$ we obtain in the shaded region values which differ little from (32):

$$\begin{aligned} \sigma(\nu_3 = 0, \nu_1 = 0.3, \nu_2 = 0.7) &= 1.24, \\ \sigma(\nu_3 = 1/3, \nu_1 = 2/3, \nu_2 = 0) &\approx 1, \\ \sigma(\nu_3 = 0.7, \nu_1 = 0.3, \nu_2 = 0) &= 0.84. \end{aligned} \quad (33)$$

With the aid of the selected scattering lengths we can duplicate also the experimental data^[8] on π -meson spectra. Thus, when $a_2 = 0.2$ and $a_0 = -0.3$ we obtain for the π^+ spectrum

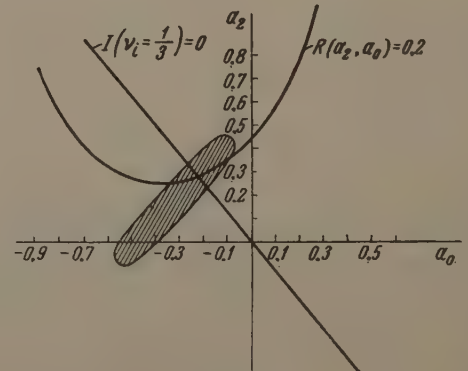


FIG. 2

$$W(t_1) = \begin{cases} 0.92 \\ 1 \\ 1.11 \end{cases} \text{ for } \begin{cases} v_1 = 0, v_2 = v_3 = 0.5, \\ v_i = 1/3, \\ v_1 = 0.7, v_2 = v_3 = 0.15. \end{cases} \quad (34)$$

Knowledge of the $\pi\pi$ scattering lengths a_2 and a_0 obtained from the τ decay enable us to draw certain conclusions regarding the relative values of the partial waves of the $\pi\pi$ interaction. From the integral equations for $\pi\pi$ scattering^[4] we can derive the following relation between a_2 and a_0 :

$$2a_0 - 5a_2 = \frac{1}{\pi} \int_0^\infty \frac{dv}{v(v+1)} \{ 2\text{Im}\Pi_0^0(v) + 9\text{Im}\Pi_1^1(v) - 5\text{Im}\Pi_0^2(v) \}. \quad (35)$$

For all pairs a_2 and a_0 duplicating the experimental τ -decay spectrum, the combination $(2a_0 - 5a_2)$ is negative. Such a result does not of necessity contradict the possibility of p-wave resonance, but in any case is evidence in favor of the existence of a ($T = 2$) resonance in $\pi\pi$ interaction.

The authors are grateful to V. N. Gribov for interesting discussions.

¹ L. B. Okun, Proc. 1960 Ann. Int. Conf. on High Energy Physics at Rochester, Univ. of Rochester, 1960, p. 743.

² S. Fubini and B. Stroffolini, Nuovo cimento 17, 263 (1960).

³ N. Khuri and S. Trieman, Phys. Rev. 119, 1115 (1960).

⁴ Hsien, Ho, and Zoellner, JETP 39, 1668 (1960), Soviet Phys. JETP 12, 1165 (1961).

⁵ M. Cini and S. Fubini, Ann. of Phys. 10, 352 (1960).

⁶ R. Sawyer and K. Wali, Phys. Rev. 119, 1429 (1960).

⁷ M. Gell-Mann and A. Rosenfeld, Annual Review of Nuclear Science 7, 1957, p. 463.

⁸ McKenna, Natali, O'Connell, Tietge, and Varshneya, Nuovo cimento 10, 763 (1958).

⁹ V. N. Gribov, JETP 41, 1221 (1961), Soviet Phys. JETP 14, in press.

Translated by J. G. Adashko
145

TRANSPORT PHENOMENA IN A PARAMAGNETIC GAS

Yu. KAGAN and L. MAKSIMOV

Submitted to JETP editor April 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 842-852 (September, 1961)

A kinetic theory is developed for a paramagnetic gas in an arbitrary magnetic field. A transport equation is derived with account of rotational degrees of freedom, discreteness of the magnetic momentum, and dependence of the scattering cross section on the angle between the direction of the molecular moment and the direction of the relative velocity, which is of decisive importance for the problem under consideration. With the thermal conductivity problem taken as an example, a general method for solving the kinetic equation is developed. An explicit expression for the thermal conductivity tensor in a magnetic field is deduced and comparison is made with the experimental results on the Senftleben effect.

1. INTRODUCTION

As is well known, the thermal conductivity of a paramagnetic gas depends on the magnetic field. This effect was first observed in oxygen by Senftleben,^[1] and later in another diatomic gas possessing paramagnetism, namely NO. In the Thirties a whole series of experimental papers was published, devoted to this effect, principally in oxygen.^[2] It was found simultaneously that the viscosity coefficient of a paramagnetic gas also depends on the magnetic field.^[3]

The experimental researches have shown that both the thermal conductivity and the viscosity decrease in a magnetic field, and this effect has a universal dependence on the ratio H/p at fixed temperature. At low values of H/p a quadratic dependence of this effect on the ratio is observed, and with further increase of H/p saturation sets in. Special experiments were also used to show that the variation of the thermal conductivity depends on the angle between the temperature gradient and the direction of the magnetic field.

A first attempt at a theoretical analysis was made by Laue.^[4] Disregarding the nature of the effect, Laue introduced in purely formal fashion the hypothesis that the mean free path depends on the angle between the direction of the molecule velocity and the direction of the magnetic field. Treating subsequently the transport phenomena in terms of the mean free path, he arrived at a phenomenological relation which contained in explicit form only the dependence on the angle between $\mathbf{H}$ and ∇T .

In another paper, Zernike and Van Lier^{[5]*} cor-

*The authors are grateful to A. Sazykin who called their attention to this paper.

rectly related this effect with the nonsphericity of the molecules. They treated the transport phenomena (thermal conductivity) in the mean-free-path approximation.

In the present paper we use the kinetic equation for molecules with rotational degrees of freedom to develop a theory for transport phenomena in a paramagnetic gas situated in a magnetic field; this theory will enable us to establish all the fundamental laws.

2. KINETIC EQUATION FOR A PARAMAGNETIC GAS IN A MAGNETIC FIELD

The effect of the magnetic field on a neutral paramagnetic gas reduces to polarization and precession of the magnetic moments of the molecules. At room temperature and reasonable magnetic fields, the polarization is very weak. Indeed, if $H \lesssim 10^4$ oe, then $\mu_0 H/kT \lesssim 10^{-3}$. The polarization thus plays no role in kinetic processes, and the effect in magnetic fields can be connected only with precession.

In the case of nonspherical molecules, the interaction (scattering cross section) will in general depend on the relative orientation of the molecules. The presence of angular-momentum precession in the magnetic field causes the orientation of the molecule relative to the direction of its velocity to vary continuously if the motion between two collisions is free. Since each orientation can, roughly speaking, be set in correspondence with its own scattering cross section, the precession results in an effective increase of the statistical weight of the larger cross sections. Consequently the kinetic coefficients will in general always decrease in a magnetic field. Obviously, in this case the decisive quantity is not $\mu_0 H/kT$, but the

ratio of the characteristic precession time to the free path time. The ratio of these times can be made without difficulty either much greater than unity or much less than unity.

It is clear from the foregoing that to describe the kinetic phenomena in a paramagnetic gas it is necessary to examine the kinetic equation for molecules having rotational degrees of freedom. We confine ourselves to an examination of linear (particularly, diatomic) molecules and to a temperature interval in which the rotational motion can be regarded classically and the vibrational degrees of freedom are not yet excited. In this case the kinetic equation can be written in the form

$$\frac{\partial f}{\partial t} + \mathbf{v} \nabla f + \frac{\partial}{\partial \mathbf{M}} (f \dot{\mathbf{M}}) = \left[\frac{\partial f}{\partial t} \right]_{\text{col}} \quad (2.1)$$

The form of the left half of (2.1) is connected with the assumption that the distribution function depends on the rotational degrees of freedom only through the angular momentum $\mathbf{M}$. In other words, we assume equiprobable distribution over the angle that characterizes the position of the molecule in a plane perpendicular to $\mathbf{M}$ (see^[6] for details).

For a complete determination of the left half of (2.1) we must determine $\dot{\mathbf{M}}$. As is well known,

$$\dot{\mathbf{M}} = [\boldsymbol{\mu} \mathbf{H}], \quad (2.2)^*$$

where $\boldsymbol{\mu}$ is the magnetic moment of the molecule. However, the magnetic moment of paramagnetic molecules is of purely quantum nature, connected with the uncompensated spin or orbital moments of the electron shell. Accordingly, the value of $\boldsymbol{\mu}$ must be determined from a quantum-mechanical analysis as the limiting value corresponding to large rotational quantum numbers.

Let the magnetic field be limited in magnitude so that the Zeeman splitting is small compared with the fine structure of the molecular spectrum. Then the magnetic moment can be determined by averaging over the unperturbed state of the molecule. In this case

$$\boldsymbol{\mu} = \overline{\boldsymbol{\mu}} = \gamma \mathbf{M}, \quad (2.3)$$

$$\gamma = \mu_0 g / \hbar. \quad (2.4)$$

Here μ_0 is the Bohr magneton, g the gyromagnetic ratio, and $\mathbf{M}$ the total momentum of the molecule. In considering large rotational quantum numbers, we shall neglect the difference between the total and rotational momentum of the molecule.

For diatomic and polyatomic linear molecules, the spin-axial interaction energy is always small,

* $[\boldsymbol{\mu} \mathbf{M}] = \boldsymbol{\mu} \times \mathbf{M}$.

at sufficiently high temperatures, compared with the energy of rotational motion (case b considered by Hund^[7]). We confine ourselves to this case only. Then, assuming that the spin of the molecule differs from zero, we can show that when $M \gg \hbar$ we have

$$\gamma \cong 2\mu_0 \sigma / M, \quad (2.5)$$

where σ runs through $2S + 1$ values

$$\sigma = -S, -S + 1, \dots, S, \quad (2.6)$$

(S is the spin of the molecule). In (2.5) we have neglected terms of order M^{-2} , and consequently, in particular, the contribution of the orbital momentum drops out.

It is clear that the distribution function in (2.1) should depend on the discrete variable σ . The last term on the left in the kinetic equation (2.1) is now written in the form

$$\gamma(\sigma) [\mathbf{M} \mathbf{H}] \frac{\partial}{\partial \mathbf{M}} f[\mathbf{r}, t, \mathbf{v}, \mathbf{M}, \sigma]. \quad (2.7)$$

Assuming that $\lambda/L \ll 1$ and $\tau_{\text{col}}/\tau \ll 1$ (λ and τ_{col} are the mean free path length and time; L and τ are the characteristic length and time for the changes in the macroscopic quantities), we seek a first-approximation solution of the kinetic equation (2.1) in the form

$$f = f^{(0)} [1 + \chi], \quad (2.8)$$

where $f^{(0)}$ is the local Maxwellian distribution function

$$f^{(0)} = n \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{1}{4\pi I kT} \exp \left(-\frac{\varepsilon}{kT} \right). \quad (2.9)$$

Here

$$\varepsilon = mu^2/2 + M^2/2I, \quad \mathbf{u} = \mathbf{v} - \mathbf{v}_0. \quad (2.10)$$

where $\mathbf{v}_0(\mathbf{r}, t)$ is the macroscopic velocity and I the moment of inertia of the linear molecule.

In (2.9) and what follows we neglect the multiplet-splitting energy (coupling of type b!) and the energy of the magnetic interaction ($\mu_0 H/kT \ll 1$). By virtue of this, in particular, $f^{(0)}$ is independent of σ .

The conditions imposed on the total distribution function are

$$\begin{aligned} \int f d\Gamma^* &= n, \quad \int f \varepsilon d\Gamma^* = \frac{5}{2} n kT, \\ \int f \mathbf{u} d\Gamma^* &= 0, \quad \int f \mathbf{M} d\Gamma^* = 0. \end{aligned} \quad (2.11)$$

Here

$$\begin{aligned} \int d\Gamma^* \dots &= \frac{1}{2S+1} \sum_{\sigma} \int d\Gamma \dots, \\ d\Gamma &= v^2 dv M dM d\Omega_M d\Omega_M. \end{aligned} \quad (2.12)$$

(The difference between the phase volume of the momentum and the ordinary volume is connected with the linearity of the molecule).

We note that in (2.11) and (2.10) we have assumed the absence of a macroscopic angular momentum.

Taking (2.9) into consideration, we arrive at the following homogeneous conditions for χ

$$\begin{aligned} \int f^{(0)} \chi d\Gamma^* &= 0, & \int f^{(0)} \chi \varepsilon d\Gamma^* &= 0, \\ \int f^{(0)} \chi u d\Gamma^* &= 0, & \int f^{(0)} \chi M d\Gamma^* &= 0. \end{aligned} \quad (2.11')$$

Let us substitute (2.8) in (2.1). Using the conservation laws and retaining the first non-vanishing terms, we obtain

$$\begin{aligned} f^{(0)} \left\{ \left(\frac{\varepsilon}{kT} - \frac{7}{2} \right) u \nabla \ln T + \frac{m}{2kT} \left(u_i u_k - \frac{1}{3} \delta_{ik} u^2 \right) \right. \\ \left. : \left(\frac{\partial v_{0k}}{\partial x_i} + \frac{\partial v_{0i}}{\partial x_k} - \frac{2}{3} \delta_{ik} \frac{\partial v_{0l}}{\partial x_l} \right) \right. \\ \left. - \left(\frac{mu^2}{3kT} - \frac{2}{5} \frac{\varepsilon}{kT} \right) \frac{\partial v_{0l}}{\partial x_l} \right\} \\ + \gamma [MH] \frac{\partial \chi}{\partial M} f^{(0)} = J_{\text{col}}(\chi), \end{aligned}$$

$$J_{\text{col}}(\chi) = \int f^{(0)} f_1^{(0)} (\chi' + \chi'_1 - \chi - \chi_1) W d\Gamma_1^* d\Gamma_1^{**}. \quad (2.13)$$

(By writing the collision integral in this form we imply that the collision probability W is an even function of the momenta of the colliding molecules.)

The form of the function W is not known in the general case. We start from the assumption that the nonsphericity of the molecules is small, and that it is difficult for the momentum of the molecule to change upon collision. Then the effect of nonsphericity manifests itself principally on the dependence of the cross section of the angle between the momenta of the molecules and the direction of the relative velocity. It is clear that this dependence should not change if $\mathbf{M}$ is replaced by $-\mathbf{M}$, and the approximate expression for W can be written

$$W d\Gamma^* d\Gamma_1^* = w g d\sigma, \quad (2.14)$$

$$\begin{aligned} w &= 1 - \lambda [P_2(\cos \hat{\mathbf{g}}\mathbf{M}) + P_2(\cos \hat{\mathbf{g}}\mathbf{M}_1) \\ &+ P_2(\cos \hat{\mathbf{g}}'\mathbf{M}) + P_2(\cos \hat{\mathbf{g}}'\mathbf{M}_1)], \end{aligned}$$

where $\mathbf{g}$ and $\mathbf{g}'$ are the relative velocities before and after collision, $d\sigma$ —differential cross section of elastic scattering of the molecules neglecting nonsphericity, and P_2 is a Legendre polynomial.

The form of the term in the square brackets is connected with the equality of the probabilities of the direct and reverse transitions, which follows from the principle of detailed balance, and also

from the necessary symmetry of W with respect to $\mathbf{M}$ and $\mathbf{M}_1$.

In writing out (2.14) we have assumed that the momentum of each molecule remains unchanged in the collision and that the interaction is independent of σ . (In addition, it is assumed that λ does not change with variation of g^2 and M^2 .) Such an interaction model is quite satisfactory in investigations of the thermal conductivity, first viscosity, and diffusion as functions of the magnetic field, but needs to be made much more precise for an analysis of second viscosity.

3. THERMAL CONDUCTIVITY OF PARAMAGNETIC GAS. THE SENFTLEBEN EFFECT

1. In the thermal conductivity problem, the expression for χ can be represented in the form

$$\chi = -\sqrt{2kT/m} \chi \nabla \ln T. \quad (3.1)$$

The corresponding kinetic equation is written

$$-u_i (u^2 + M^2 - 7/2) f^{(0)} + \gamma f^{(0)} [MH] \partial \chi_i / \partial M = J_{\text{col}}(\chi_i) \quad (3.2)$$

Starting from this expression, we shall take $\mathbf{u}$ and $\mathbf{M}$ to mean everywhere dimensionless quantities that differ from the preceding quantities by the factors $\sqrt{m/2kT}$ and $\sqrt{1/2IkT}$, respectively.

We expand the function χ in irreducible tensors made up of $\mathbf{u}$ and $\mathbf{M}$:

$$\begin{aligned} \chi_i(\mathbf{u}, \mathbf{M}, \sigma) \\ = \sum_{p, q} T_{ik_1 \dots k_{p+q}}^{pq} [u_{k_1} \dots u_{k_p}] [M_{k_{p+1}} \dots M_{k_{p+q}}], \end{aligned} \quad (3.3)$$

where the square brackets denote the corresponding irreducible tensors. The expansion coefficients (3.3) are functions of u^2 , M^2 , and of the index σ , and can in turn be expanded in the orthogonal polynomials

$$T_{ik_1 \dots k_{p+q}}^{pq} = \sum_{st} T_{ik_1 \dots k_{p+q}}^{pqst} S_{2p+1}^{(s)}(u^2) S_{2q}^{(t)}(M^2), \quad (3.3a)$$

where $S_n^{(m)}$ are Sonin polynomials of index n . The coefficients in the expansion (3.3a) depend only on the discrete variable σ .

From the form of (3.2) it follows that χ is an odd function in $\mathbf{u}$ and an even function in $\mathbf{M}$. This means that the only nonvanishing coefficients in (3.3) are those with odd p and even q .

The conditions (2.11') leads to the single relationship

$$T_{ik}^{1000} = 0.$$

Let us write down the equation for the heat flow

$$q_i \equiv -\chi_{ik} (\nabla T)_k = kT \sqrt{\frac{2kT}{m}} \int f^{(0)} \chi u_i (u^2 + M^2) d\Gamma^*.$$

Using (3.1), (3.3), and (3.3a) we obtain for the thermal conductivity tensor, in the general case,

$$\kappa_{ik} = k(2kT/m) \left(\frac{5}{4} \bar{T}_{ki}^{1010} + \frac{1}{2} \bar{T}_{ki}^{1001} \right),$$

$$\bar{T} = \frac{1}{2S+1} \sum_{\sigma} T(\sigma). \quad (3.4)$$

Thus, the thermal conductivity tensor is determined in terms of two coefficients of the expansion (3.3) averaged over σ .

Let us proceed to find the approximate solution of (3.2). For this purpose we confine ourselves in (3.3) to terms with $p = 1$ and $q \leq 2$, and in (3.3a) we retain the first nonvanishing terms. Then

$$\chi_i = T_{ik}^1 \psi_k^1 + T_{ik}^2 \psi_k^2 + T_{iklm}^3 \psi_{klm}^3, \quad (3.5)$$

$$\psi_k^1 = u_k(u^2 - \frac{5}{2}), \quad \psi_k^2 = u_k(M^2 - 1),$$

$$\psi_{klm}^3 = u_k(M_l M_m - \frac{1}{3} \delta_{lm} M^2), \quad (3.6)$$

$$T^1 = T^{1010}, \quad T^2 = T^{1001}, \quad T^3 = T^{1200}. \quad (3.7)$$

If we substitute (3.3) into (3.2), we can readily verify that the only components of (3.3) contributing to the term containing the magnetic field are those with $q \neq 0$. On the other hand, a detailed analysis shows that an account of the terms with $q > 0$ and $p > 1$ in (3.3) yields for the thermal conductivity corrections that are proportional to the nonsphericity parameter raised to the power $q + 2(p - 1)$. Since the sphericity is assumed from the very outset to be small, the limitation (3.5) concerning p and q is of an obvious character [the approximation (3.5) is actually quadratic in λ *].

The fact that terms with high values of s and t are discarded in (3.5) is not connected with any explicit smallness parameter. Nonetheless, as is well known, it usually turns out that these terms are numerically small. We shall later on (Sec. 4) refine (3.5), taking account of the terms of next higher order in t for $q = 2$.

2. To find the coefficients in (3.5) we multiply (3.2) by each of the functions (3.6) and integrate over the phase volume (without averaging over the discrete variable σ [see (2.13)]. As a result we arrive at the system

$$\frac{5}{4} n \delta_{ik} = T_{ik}^1 A^{11} + \bar{T}_{ik}^1 B^{11} + 2(T_{illk}^3 - \bar{T}_{illk}^3) A^{13},$$

$$\frac{1}{2} n \delta_{ik} = T_{ik}^2 A^{22} + 2T_{illk}^3 A^{23},$$

$$-\xi T_{iklm} H_{lmnp} = (T_{il}^1 - \bar{T}_{il}^1) \Delta_{iknp} A^{13}$$

$$+ T_{il}^2 \Delta_{iknp} A^{23} + 2T_{iknp}^3 A^{33}. \quad (3.8)$$

*We note that actually the nonsphericity parameter employed is not λ [see (2.14)], but at least $\lambda/5$.

Here

$$\Delta_{iklm} = -\frac{2}{3} \delta_{ik} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{im} \delta_{kl},$$

$$H_{iklm} = h_{il} \delta_{km} + h_{im} \delta_{kl} + h_{kl} \delta_{im} + h_{km} \delta_{il},$$

$$h_{ik} = \varepsilon_{ikl} H_l / H,$$

$$\xi(\sigma) = n(\sqrt{\pi} \mu_0 / 20 \sqrt{2kTl}) H \sigma, \quad (3.9)$$

where ε_{ikl} is a unit antisymmetrical tensor.

In the calculation of the term with the magnetic field in (3.8) we took account of the dependence of the coefficient on the rotational momentum M [see (2.5)]. An explicit expression for $A^{\alpha\beta}$ and $B^{\alpha\beta}$ is given in Appendix A.

3. We present the solutions of (3.8) in the form

$$T^{\alpha} = X^{\alpha} + Y^{\alpha}(H), \quad (3.10)$$

where X^{α} is independent of H and $Y^{\alpha}(0) = 0$.

We consider first the case $H = 0$. We seek the tensors X^{α} in the form

$$X_{ik}^1 = X_1 \delta_{ik}, \quad X_{ik}^2 = X_2 \delta_{ik}, \quad X_{iklm}^3 = X_3 \Delta_{iklm}.$$

(In the absence of a magnetic field, $T^{\alpha} = X^{\alpha}$ are independent of the discrete variable.) From (3.8) we obtain

$$X_1 = \frac{5}{4} \frac{n}{A^{11} + B^{11}}, \quad X_2 = \frac{1}{2} \frac{n}{A^{22}} \left[1 - \frac{10}{3} \frac{(A^{23})^2}{A^{22} A^{33}} \right]^{-1},$$

$$X_3 = -A^{23} X_2 / 2A^{33}. \quad (3.11)$$

The coefficients X_1 and X_2 determine the thermal-conductivity tensor in the absence of a magnetic field [see (3.4)]:

$$(\kappa_{ik})_{H=0} = \kappa_0 \delta_{ik}, \quad \kappa_0 = nk(2kT/m) \left(\frac{5}{4} X_1 + \frac{1}{2} X_2 \right). \quad (3.4')$$

4. Let now the magnetic field differ from zero. Substituting (3.10) into (3.8) we arrive at a system of equations for Y^{α} :

$$0 = Y_{ik}^1 A^{11} + \bar{Y}_{ik}^1 B^{11} + 2(Y_{illk}^3 - \bar{Y}_{illk}^3) A^{13},$$

$$0 = Y_{ik}^2 A^{22} + 2Y_{illk}^3 A^{23}. \quad (3.12)$$

As shown in Appendix A, when $\lambda \ll 1$ the nondiagonal elements of the matrix $A^{\alpha\beta}$ are small compared with the diagonal elements. From the first two equations of (3.12) we can conclude that $Y_{ik}^2 \sim Y_{illk}^3 A^{23}/A^{22}$ and $Y_{ik}^1 \sim Y_{illk}^3 A^{13}/A^{11}$. We can certainly then neglect in (3.12) the first two terms on the right side. We obtain thus an equation for Y_{iklm}^3 in closed form:

$$Y_{iknp}^3 A^{33} + \frac{1}{2} \xi Y_{iklm}^3 H_{lmnp} = -\xi X_3 H_{iknp}. \quad (3.13)$$

The thermal-conductivity tensor (3.8) includes the tensors Y_{ik}^1 and Y_{ik}^2 , averaged over the discrete variable. Averaging the first two equations in (3.12) over σ , we obtain

$$\bar{Y}_{ik}^1 = 0, \quad \bar{Y}_{ik}^2 = -2A^{23}\bar{Y}_{ilk}^3/A^{22}. \quad (3.14)$$

Thus, the thermal conductivity depends only on that part of the tensor Y_{iklm}^3 which is even in σ . Taking this into account, we seek a solution of (3.13) by writing for the tensor Y_{iklm}^3

$$Y^3 = Y^+ + Y^-, \quad (3.15)$$

where Y^+ and Y^- are respectively even and odd in σ .

Starting with (3.15), we shall use the convenient symbol L to denote fourth-rank tensors, and the product of two tensors LM will again be taken to be a tensor of fourth rank with components

$$L_{iklm}M_{lmnp}. \quad (3.16)$$

Using this notation, we rewrite (3.13) in the form

$$A^{33}(Y^+ + Y^-) + \frac{1}{2}\xi(Y^+ + Y^-)H = -\xi X_3 H.$$

This equation separates into two:

$$A^{33}Y^+ + \frac{1}{2}\xi Y^+ H = 0, \quad (3.17)$$

$$A^{33}Y^- + \frac{1}{2}\xi Y^- H = -\xi X_3 H. \quad (3.18)$$

We took account here of the fact that ξ is a linear function of the variable σ [see (3.9)].

Multiplying (3.18) from the right by ξH and using (3.17) we obtain an equation for Y^+ in closed form:

$$-(A^{33})^2 Y^+ + \frac{1}{4}\xi^2 Y^+ H^2 = -\frac{1}{2}\xi^2 X_3 H^2. \quad (3.19)$$

It can be shown (see Appendix B) that

$$H^6 = -4(5H^4 + 16H^2). \quad (3.20)$$

Using this equation to solve (3.19), we get

$$Y^+ = c_1 H^2 + c_2 H^4, \quad (3.21)$$

$$c_1 = \frac{\eta^2(1+5\eta^2)}{2(1+\eta^2)(1+4\eta^2)} X_3,$$

$$c_2 = \frac{\eta^4}{8(1+\eta^2)(1+4\eta^2)} X_3, \quad (3.22)$$

$$\eta = \xi/A^{33}. \quad (3.23)$$

According to (3.14) we must contract Y^+ about the two internal indices. Using the contractions of the tensors H^2 and H^4 , written out in the Appendix B, we can readily obtain

$$Y_{ilk}^+ = (c_1 \delta_{ik} + c_2 H_i H_k / H^2) X_3, \quad (3.24)$$

$$c_1' = -\frac{3\eta^2(3+4\eta^2)}{(1+\eta^2)(1+4\eta^2)}, \quad c_2' = \frac{\eta^2(7+4\eta^2)}{(1+\eta^2)(1+4\eta^2)}. \quad (3.25)$$

The expression (3.24) is the solution of our problem. Indeed, substituting (3.24) into (3.14) and using (3.4), (3.7), (3.10), and (3.11) we obtain directly a general expression for the thermal-con-

ductivity tensor. Separating from this tensor only the part dependent on the magnetic field, and taking (3.4') into account, we get

$$\Delta \kappa_{ik} = \kappa_{ik} - (\kappa_{ik})_{H=0} = \kappa_0 \frac{\psi}{2S+1} \sum_{\sigma} \left(c_1' \delta_{ik} + c_2' \frac{H_i H_k}{H^2} \right), \quad (3.26)$$

$$\psi = \frac{(A^{23})^2}{A^{22}A^{33}} \left\{ 1 + 5 \frac{A^{22}}{A^{11} + B^{22}} \left[1 - \frac{10}{3} \frac{(A^{23})^2}{A^{22}A^{33}} \right] \right\}^{-1}. \quad (3.27)$$

The expressions in (3.26), (3.27), and (3.25) describe the Senftleben effect completely.

4. DISCUSSION OF RESULTS

The expressions obtained show that in a paramagnetic gas in the presence of a magnetic field, the coefficient of thermal conductivity becomes anisotropic. This causes the thermal conductivity to have different values when the magnetic field is parallel and perpendicular to the temperature gradient. Moreover, when the angle between the vectors H and ∇T differs from 0 to 90°, a heat-flow component exists in a direction perpendicular to the temperature gradient.

The character of the dependence of the tensor $\Delta \kappa_{ik}$ on the magnetic field and on the pressure is completely determined by (3.25), where the value of η is [see (3.22) and (3.9)]

$$\eta = \frac{\sqrt{\pi}}{20} \frac{\mu_0 k n^2 \sqrt{T}}{\sqrt{2kl} A^{33}} \frac{H}{p} \sigma. \quad (4.1)$$

It follows from (4.1) that at a fixed temperature the effect depends only on the ratio H/p , which coincides with the main experimental result obtained in [2]. The temperature dependence at fixed H/p is determined in (4.1) by the ratio $n^2 \sqrt{T}/A^{33}$.

It is shown in Appendix A that $A^{33} \sim \Omega^{11} n^2$ and consequently the temperature dependence is essentially connected with the form of the molecule scattering cross section. Thus, for the solid-sphere model $\Omega^{11} \sim T^{1/2}$ and (4.1) is completely independent of T . In the case of Maxwellian molecules ($U \sim r^{-4}$), Ω^{11} is independent of the temperature and $\eta \sim T^{1/2}$. The true character of the interaction will lead, as a rule, to an intermediate dependence, that is, $\sim T^n$, where $0 \leq n \leq 1/2$.

The overwhelming majority of the experiments on the determination of the thermal conductivity of a paramagnetic gas were carried out with oxygen. From the temperature dependence of the ordinary ($H = 0$) kinetic coefficients for O_2 (see, for example, [8]), we can conclude that Ω^{11} is independent of T . Consequently, the character of relative variation of $\Delta \kappa_{ik}$ should depend in prac-

tice on the combination $H\sqrt{T}/p$, as was indeed observed experimentally.^[2]

Let us examine the variations of the thermal-conductivity coefficients in the case when H is parallel and perpendicular to ∇T .

From (3.26) we obtain

$$\frac{\Delta\kappa_{\perp}}{\kappa_0} = \frac{\psi}{2S+1} \sum_{\sigma} \tilde{c}_{\sigma}, \quad \frac{\Delta\kappa_{\parallel}}{\kappa_0} = \frac{\psi}{2S+1} \sum_{\sigma} (c'_{\sigma} + c''_{\sigma}). \quad (4.2)$$

From (3.27) and (3.25) it follows that at all values of η

$$\Delta\kappa_{\perp}/\kappa_0 < 0, \quad \Delta\kappa_{\parallel}/\kappa_0 < 0. \quad (4.3)$$

At sufficiently large values saturation sets in, and both quantities attain the following absolute maxima

$$\left(\frac{\Delta\kappa_{\perp}}{\kappa_0} \right)_{\infty} = -\frac{2S}{2S+1} 3\psi, \quad \left(\frac{\Delta\kappa_{\parallel}}{\kappa_0} \right)_{\infty} = -\frac{2S}{2S+1} 2\psi. \quad (4.4)$$

Hence

$$(\Delta\kappa_{\parallel}/\Delta\kappa_{\perp})_{\infty} = 2/3, \quad (4.5)$$

which is in splendid agreement with Senftleben's experiments.

From expressions (3.26) and (3.27), and also from formulas (A.5) of Appendix A, it can be concluded that the change in the thermal conductivity in a magnetic field is completely connected with the nonsphericity of the interaction, and that $\Delta\kappa_{ik} \sim \lambda^2$. If we use (4.3), we can readily relate the parameter with the value $(\Delta\kappa/\kappa_0)_{\infty}$ which is independent of the magnetic field and of the pressure. Comparison with experiments shows that in the case of oxygen λ is close to unity. As already indicated, the actual expansion parameter is a quantity smaller than $\lambda/5$, and all the assumptions made above remain in force.

Let us examine the region of small values of η . Then

$$\left(\frac{\Delta\kappa_{\perp}}{\kappa_0} \right)_{H \rightarrow 0} = -\frac{9\psi}{2S+1} \sum_{\sigma} \eta^2, \quad \left(\frac{\Delta\kappa_{\parallel}}{\kappa_0} \right)_{H \rightarrow 0} = -\frac{2\psi}{2S+1} \sum_{\sigma} \eta^2. \quad (4.6)$$

In the case of oxygen $S = 1$ and (4.6) goes into

$$(\Delta\kappa_{\perp}/\kappa_0)_{H \rightarrow 0} = -6\psi\eta_1^2, \quad (\Delta\kappa_{\parallel}/\kappa_0)_{H \rightarrow 0} = -\frac{4}{3}\psi\eta_1^2, \quad (4.6')$$

where $\eta_1 = (\eta)_{\sigma=1}$.

The dependence of (4.6') on the magnetic field and on the pressure coincides with the experi-

mentally observed one. Bearing in mind a comparison of the absolute values of the calculated and measured coefficients, we have increased the number of terms in (3.5), taking in (3.3a) the terms of higher order in M^2 with $q = 2$ (specifically the terms responsible for the effect and connected with the magnetic field). The term with $t = 1$ [see (3.3a)] introduces a correction which is still noticeable, whereas inclusion of $t = 2$ changes the results very little. The calculation procedure obviously remains the same, and we shall only give the final results.

Inclusion of the term with $t = 1$ changes the numerical coefficients in (4.4) by a factor $4/3$, and a multiplier $25/36$ appears in the coefficients from (4.6). We then have for the ratio $(\Delta\kappa_{\perp})_{H \rightarrow 0}/(\Delta\kappa_{\perp})_{\infty}$ a value

$$(\Delta\kappa_{\perp})_{H \rightarrow 0}/(\Delta\kappa_{\perp})_{\infty} = 25\eta^2/16 = a^2 (H/p)^2.$$

We have calculated a for oxygen, using the well known value $\hbar^2/2Ik = 2.1^\circ \text{K}$ and determining Ω^{11} from the value of the self-diffusion coefficient.^[8] If H is in gauss and p in mm Hg, we obtain $a \approx 2.5$. The corresponding experimental value is close to 1.7. If we recognize that the number of terms included in (3.3a) is small, and if we bear in mind in particular the differential term in (3.2), the agreement of the results should be regarded as satisfactory.

APPENDIX A

Let us determine the coefficients in the right half of the system (3.8). We integrate the collision integral with one of the functions ψ^{α} (3.6):*

$$-\int d\Gamma \psi_k^{\alpha} J_{\text{col}}(\chi_i) = -\langle \psi_k^{\alpha} (\chi'_i + \bar{\chi}_{1i} - \chi_i - \bar{\chi}_{1i}) \rangle. \quad (A.1)$$

We introduce here the symbol $\langle \Phi \rangle$ for the integral

$$\int \sqrt{\frac{4kT}{m}} d\Gamma d\Gamma_1 \hat{f}_1^{(0)} \hat{f}_1^{(0)} \omega g d\sigma d\Omega, \quad (A.2)$$

where g is a dimensionless relative velocity. In (A.1) we averaged in explicit form over the discrete variable of particle 1. Using the well known symmetry properties of the integrand of (A.2) we can transform the integral of (A.1) to the form

$$\sum_{\rho} T_{it}^{\rho} A_{kl}^{\alpha\beta} + \sum_{\beta} \bar{T}_{it}^{\beta} B_{kl}^{\alpha\beta}, \quad (A.3)$$

$$A_{kl}^{\alpha\beta} = \frac{1}{2} \langle [(\psi_k^{\alpha})' - \psi_k^{\alpha}] [(\psi_l^{\beta})' - \psi_l^{\beta}] \rangle, \quad B_{kl}^{\alpha\beta} = \frac{1}{2} \langle [(\psi_k^{\alpha})' - \psi_k^{\alpha}] [(\psi_l^{\beta})_1 - (\psi_l^{\beta})_1] \rangle, \quad (A.4)$$

*It must be remembered that the function ψ^3 has three indices

$$\psi_{\mathbf{k}}^3 \equiv \psi_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3}^3.$$

with

$$A_{kl}^{\alpha\beta} = A_{lk}^{\beta\alpha}, \quad B_{kl}^{\alpha\beta} = B_{lk}^{\beta\alpha}.$$

The coefficients of (A.4) can be directly calculated for the model of (2.14):

$$A_{ik}^{11} = A^{11}\delta_{ik},$$

$$A^{11} = \frac{1}{3}n^2\left(\frac{55}{4}\Omega^{11} - 5\Omega^{12} + \Omega^{13} + 2\Omega^{22}\right),$$

$$B_{ik}^{11} = B^{11}\delta_{ik},$$

$$B^{11} = \frac{1}{3}n^2\left(-\frac{55}{4}\Omega^{11} + 5\Omega^{12} - \Omega^{13} + 2\Omega^{22}\right),$$

$$A_{ik}^{12} = B_{ik}^{12} = B_{ik}^{22} = 0, \quad A_{ik}^{22} = A^{22}\delta_{ik}, \quad A^{22} = \frac{4}{3}n^2\Omega^{11};$$

$$A_{ik_1k_2k_3}^{13} = -B_{ik_1k_2k_3}^{13} = A^{13}\Delta_{ik_1k_2k_3},$$

$$A^{13} = \frac{1}{25}\lambda n^2\left[\frac{4}{3}\left(-\frac{7}{2}\Omega^{11} + \Omega^{12}\right) + \left(\frac{5}{2}\Omega^{21} - \Omega^{22}\right)\right],$$

$$A_{ik_1k_2k_3}^{23} = A^{23}\Delta_{ik_1k_2k_3}, \quad A^{23} = \frac{2}{25}\lambda n^2\left(\frac{4}{3}\Omega^{11} - \Omega^{21}\right),$$

$$B_{ik_1k_2k_3}^{23} = 0, \quad A_{ik_1k_2k_3}^{33} = A^{33}\left(\delta_{ik_1}\Delta_{ik_2k_3} + \frac{1}{35}\lambda D_{ik_1k_2k_3}\right),$$

$$A^{33} = \frac{2}{15}A^{22}. \quad (\text{A.5})$$

The numerical coefficients in (A.5) are determined in terms of standard integrals of the kinetic theory of monatomic gases (see^[8]),

$$\Omega^{(l,s)} = \sqrt{\frac{kT}{\pi m}} \int e^{-g^2} g^{2s+3} (1 - \cos^l \theta) d\sigma dg. \quad (\text{A.6})$$

The tensor Δ_{iklm} has been defined in the text [see (3.9)]. The sixth-rank tensor D_{hiklmn} is symmetrical within the index pairs hi , kl , mn , and vanishes upon contraction in any of these pairs.

The tensor $A_{i_1i_2i_3k_1k_2k_3}^{33}$ makes the following contribution to (A.3):

$$\begin{aligned} & T_{i_1i_2i_3}^{33} A_{k_1k_2k_3}^{33} \\ &= 2A^{33} \left\{ T_{ik_1k_2k_3}^3 + \frac{1}{35}\lambda \left[-\frac{4}{3}T_{ik_1k_2k_3} + T_{ik_2k_3k_1} + T_{ik_3k_1k_2} \right. \right. \\ & \quad \left. \left. - \frac{4}{3}T_{illk_1}\delta_{k_2k_3} + T_{illk_2}\delta_{k_1k_3} + T_{illk_3}\delta_{k_1k_2} \right] \right\}. \quad (\text{A.7}) \end{aligned}$$

We have neglected in the text terms proportional to $\lambda/35$ in (A.7).

APPENDIX B

The powers of the tensor H can be readily calculated with the aid of tensors of the form

$$\begin{aligned} F_{iklm}^{\alpha\beta} &\equiv F(f_{ik}^\alpha, f_{lm}^\beta) = f_{il}^\alpha f_{km}^\beta + f_{im}^\alpha f_{kl}^\beta + f_{kl}^\alpha f_{im}^\beta + f_{km}^\alpha f_{il}^\beta, \\ \alpha, \beta &= 0, 1, 2, \end{aligned} \quad (\text{B.1})$$

where

$$f_{ik}^0 = \delta_{ik}, \quad f_{ik}^1 = h_{ik}, \quad f_{ik}^2 = h_{il}h_{ik} = -\delta_{ik} + H_i H_k / H^2. \quad (\text{B.2})$$

The tensors (B.1) have the symmetry properties

$$F_{iklm}^{\alpha\beta} = F_{kilm}^{\alpha\beta} = F_{ikml}^{\alpha\beta} = F_{iklm}^{\beta\alpha}. \quad (\text{B.3})$$

The product of two tensors given by (B.1) is

$$F_{ikst}^{\alpha\beta} F_{sltm}^{\gamma\delta} = 2[F(f_{is}^\alpha f_{sk}^\gamma, f_{lt}^\beta f_{tm}^\delta) + F(f_{is}^\alpha f_{sk}^\delta, f_{lt}^\beta f_{tm}^\gamma)]. \quad (\text{B.4})$$

Using (B.4), we obtain

$$H = F^{01}, \quad H^2 = 2(F^{02} + F^{11}),$$

$$H^4 = -4H^2 + 24(F^{22} - F^{11}),$$

$$H^6 = -4(5H^4 + 16H^2). \quad (\text{B.5})$$

The thermal-conductivity tensor includes the tensors H^2 and H^4 , contracted in the internal indices

$$(H^2)_{illk} = 2(-9\delta_{ik} + 7H_i H_k / H^2),$$

$$(H^4)_{illk} = 8(33\delta_{ik} - 31H_i H_k / H^2). \quad (\text{B.6})$$

In calculations with more terms than in (3.5), it is best to use instead of powers of the tensor H the tensors

$$G_1 = F^{02} + F^{22}, \quad G_2 = \frac{1}{4}(F^{11} - F^{22}),$$

which are the eigenfunctions of the operator H^2 :

$$\frac{1}{4}G_1 H^2 = -G_1, \quad \frac{1}{4}G_2 H^2 = -4G_2.$$

The tensor H^2 is connected with these tensors by the relation

$$H^2 = G_1 + 4G_2.$$

¹ H. Senftleben, *Physik. Z.* **31**, 961 (1930).

² H. Senftleben and J. Piezner, *Ann. Physik* **16**, 907 (1933); **27**, 108 and 117 (1936); **30**, 541 (1937). E. Rieger, *Ann. Physik* **31**, 453 (1938). H. Torwegge, *Ann. Physik* **33**, 459 (1938).

³ H. Senftleben and H. Gladisch, *Ann. Physik* **30**, 713 (1937); **33**, 471 (1938). H. Engelhardt and H. Sach, *Physik. Z.* **33**, 724 (1932). M. Trantz and E. Froeschel, *Physik. Z.* **33**, 947 (1932).

⁴ M. Laue, *Ann. Physik* **23**, 1 (1935); **26**, 373 (1936).

⁵ F. Zernike and C. Van Lier, *Physica* **6**, 961 (1939).

⁶ Yu. Kagan and A. Afanas'ev, *JETP* **41**, 1536 (1961), *Soviet Phys. JETP* **14**, in press.

⁷ L. Landau and E. Lifshitz, *Quantum Mechanics*, Pergamon, 1958.

⁸ Hirschfelder, Curtiss, and Bird, *Molecular Theory of Gases and Liquids*.

DETERMINATION OF THE HYPERFINE SPLITTING ENERGY OF THE $1s$ MUONIUM STATE

A. O. VAISENBERG

Institute of Theoretical and Experimental Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 853-855 (September, 1961)

The hyperfine splitting of the ground state of muonium is estimated on the basis of the available experimental data on the dependence of μ^+ polarization in nuclear emulsion on the magnetic field.

THE aim of the present work is to obtain an estimate of the hyperfine splitting in the $1s$ state of muonium from the existing data on the dependence of the polarization of positive muons in nuclear emulsion on the strength of the applied magnetic field.^[1-3]

It is well known that the main mechanism for depolarization of positive muons in matter is the formation of muonium atoms (μ^+ , e^-) in the $1s$ state. An external magnetic field that is strong enough to break the connection between the magnetic moments of the muon and electron eliminates the depolarization, which arises from the hyperfine splitting in this state. The theoretical treatment^[4,5] of this problem shows that the polarization P depends on the applied field H according to

$$P = 3a = P_0 \left(\frac{1}{2} + \frac{x^2}{2(1+x^2)} \right), \quad x = \frac{H}{H_0}. \quad (1)$$

Here a is the asymmetry coefficient of the spatial distribution of electrons in $\mu^+ \rightarrow e^+$ decay, P_0 is the muon polarization at the instant of decay, and H_0 characterizes the average field in the first electronic Bohr orbit in muonium due to the muon magnetic moment. H_0 is related to the hyperfine splitting ΔE of the $1s$ state:

$$H_0 = \Delta E / (-2\mu_e + 2\mu_\mu) \quad (2)$$

where μ_e and μ_μ are the magnetic moments of the electron and muon respectively. ΔE is given by the well-known Fermi formula

$$\Delta E \cong 32\mu_e\mu_\mu / 3a_B^3, \quad (3)$$

where a_B is the radius of the first Bohr orbit in muonium. If the magnetic moment values $\mu_e = e\hbar/2m_e c$ and $\mu_\mu = e\hbar/2m_\mu c$ are substituted into (2) and (3), we obtain $H_0 = 1580$ gauss. Taking into account radiative corrections and the effect of the reduced mass changes this value to $H_0 = 1588$ gauss. take into account the fact that some of the muons

Actually, when muons are stopped in emulsion^[1,2] or other substances^[7] the growth of polarization with magnetic field does not obey Eq. (1) because of the charge exchange which occurs when a nearly-spent muon which is still sufficiently fast loses and captures electrons several (n) times. Each new electron capture further "dilutes" the polarization; therefore, for a given value of the field H the value of the polarization P is less than that predicted by Eq. (1). Ferrell, Lee, and Pal^[8] obtained a formula which takes into account the charge exchange near the end of the muon path:

$$P = P_0 \left[1 - \frac{1}{2} \left(\frac{1}{1 + \tau^{-2} + x^2} \right) \right]^n. \quad (4)$$

Here $\tau/2$ is the average lifetime of the muonium (in units of $\hbar/\Delta E = 3.6 \times 10^{-11}$ sec) in each capture. If only one capture occurs ($n = 1$, $\tau = \infty$), this formula becomes the same as Eq. (1).

Rather accurate measurements have been made^[1-3] of the asymmetry parameter $a = P/3$ in nuclear emulsion placed in a magnetic field. All of the experiments were performed by nearly identical methods and similar criteria for selecting events were used. The consistent group of data thus obtained is shown in Table I. Assuming the depolarization mechanism described by Eq. (3), we can use the data of Table I to evaluate the depolarization parameters n and τ and the hyperfine splitting ΔE .

The problem consists of finding the minimum of the function

$$\chi^2 = \sum (\Delta a_i)^2 / \sigma_i^2$$

under simultaneous variation of the parameters τ , n , and H_0 , i.e., of solving the system of equations

$$\partial \chi^2 / \partial \tau = \partial \chi^2 / \partial n = \partial \chi^2 / \partial H_0 = 0. \quad (5)$$

In the analysis of the data it is necessary to

Table I

Reference	H, gauss	-a	Reference	H, gauss	-a
[1]	0	0.095 ± 0.009	[1]	2500	0.206 ± 0.020
[1]	54	0.117 ± 0.020	[1]	3500	0.185 ± 0.020
[1]	110	0.110 ± 0.020	[1]	5100	0.196 ± 0.020
[1]	206	0.093 ± 0.020	[1]	6300	0.265 ± 0.021
[2]	405	0.114 ± 0.021	[2]	10000	0.263 ± 0.012
[1]	420	0.142 ± 0.020	[1]	14000	0.262 ± 0.028
[1]	680	0.149 ± 0.017	[1]	17000	0.280 ± 0.014
[2]	805	0.096 ± 0.020	[2]	20000	0.272 ± 0.023
[1]	1300	0.142 ± 0.017	[2, 3]	25000	0.290 ± 0.013
[2]	1610	0.178 ± 0.020	[2]	27000	0.284 ± 0.016
[1]	1900	0.155 ± 0.022	[2]	35000	0.290 ± 0.013
[2]	2370	0.184 ± 0.020			

stop in the gelatin, where the depolarization is either small or absent, and some stop in the AgBr crystals which are responsible for most of the depolarization.^[9] If we let f be the ratio of the number of stoppings in the gelatin to the total number of muon stoppings in the emulsion, then we obtain

$$P = 3a = P_0 \left[f + (1 - f) \left(1 - \frac{0.5}{1 - \tau^2 + \chi^2} \right)^n \right] \quad (6)$$

which is to be used in the analysis of the data in Table I. As the most likely value of f we choose $f = 0.32$,^[1] which is obtained as the ratio of the asymmetry coefficient $a(0)$ for $H = 0$ to the asymmetry coefficient $a(\infty)$ obtained by averaging the data in Table I for $H \geq 17$ kilogauss:

$f = (0.095 \pm 0.009) / (0.292 \pm 0.007)$. P_0 in (6) is taken to be $P_0 = 3a(\infty) = 0.88 \pm 0.02$.

The solution of (5) obtained on an electronic computer by the method of "gradient descent" is $\tau = 1.22$, $n = 6.90$, and $H_0 = 1560$ gauss, with $\chi^2 = 28$. Since the number of degrees of freedom is 22, the result for χ^2 is rather good.

The weak links in this determination of H_0 and thence, via Eq. (2), the hyperfine splitting ΔE in the 1s state of muonium are first, the use of a specific mechanism for the dilution of polarization which occurs in the slowing-down process, and second, the value of f the fraction of stoppings in the gelatin. It can be shown, however, that all values of f in a wide range give the same value of H_0 .

Table II shows the solutions we obtained for n , τ , and H_0 for values of f in the interval 0–0.5.

We see that a radical change in f leads to noticeable changes in the parameters n and τ but does not change the value of H_0 nor, consequently, the value of ΔE . The statistical accuracy

Table II

f	χ^2	n	τ	H_0
0	26.2	5.0	0.73	1560
0.20	26.8	6.1	0.88	1560
0.33	28.2	6.9	1.22	1560
0.40	32.1	7.4	2.50	1560
0.50	65.0	12.1	2.58	1560

of our determination of ΔE , as evaluated by the usual method of constructing the error matrix,^[10] is about 10%.

Thus, the value we obtain for the hyperfine splitting of the 1s state of muonium agrees, within the limits of error, with the theoretical value.

Measurements analogous in principle to these were carried out by Prepost, Hughes, and Ziock^[11] on muons stopping in argon. Their experiment, performed with electronic counters, showed that the observed increase of polarization with increasing field was consistent with the theoretical value of ΔE . Their measurements were not sufficiently accurate to allow them to make a quantitative evaluation of ΔE .

The author thanks V. N. Petrukhin for carrying out the calculation on the electronic computer.

¹A. O. Vaĭsenberg and V. A. Smirnit-skii, JETP 39, 242 (1960), Soviet Phys. JETP 12, 175 (1961).

²Gurevich, Nikol'skii, and Ali-Zade, JETP 40, 452 (1961), Soviet Phys. 13, 313 (1961).

³Lynch, Orear, and Rosendorff, Phys. Rev. 118, 284 (1960).

⁴R. A. Ferrell and F. Chaos, Phys. Rev. 107, 1322 (1957).

⁵G. Breit and V. W. Hughes, Phys. Rev. 106, 1293 (1957).

⁶H. Bethe and E. Salpeter, Quantum Mechanics of One- and Two-electron Systems, Springer, Berlin 1957.

⁷Sens, Swanson, Telegdi, and Yovanovich, Phys. Rev. 107, 1465 (1957).

⁸Ferrell, Lee, and Pal, Phys. Rev. 118, 317 (1960); 119, 2098 (E) (1960).

⁹L. M. Lederman, Proceedings of the International Conference on High-Energy Physics, Rochester 1957.

¹⁰J. Orear, Lectures on Statistics for Physicists, Dubna 1959.

¹¹Prepost, Hughes, and Ziock, Phys. Rev. Lett. 6, 19 (1961).

Translated by M. Bolsterli
147

π^-p INTERACTION AT 7 BEV

I. M. GRAMENITSKII, I. M. DREMIN, and D. S. CHERNAVSKII

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R., Joint Institute for Nuclear Research

Submitted to JETP editor April 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 856-860 (September, 1961)

A method developed earlier^[3] is employed to describe the π^-p interaction at 7 Bev. The results of the calculations are compared with the experimental data. The comparison shows that the main characteristics of recoil nucleons (their energy spectrum, angular distribution, etc.) satisfactorily agree with the calculations. The problem of obtaining information on the $\pi\pi$ -interaction cross section at energies for which it is mainly inelastic and, in particular, on the $\pi\pi$ diffraction cross section is discussed. It is shown that suitable events can be chosen from the experimental material on the π^-p interaction and can be used for determining the $\pi\pi$ diffraction cross section.

EXPERIMENTAL data have recently been obtained on the π^-p interaction at a laboratory energy $E_L = 7$ Bev.^[1] Some of the data (for example, the angular and energy distributions of the nucleons) indicate that peripheral interactions play an important role.

In this connection, we studied this process in the one-meson approximation on the basis of the diagram shown in Fig. 1. This method has already been used for the calculation of the πp interaction at 5 Bev^{[2]*} and of the nucleon-nucleon interaction at 9 Bev.^[3] The conditions of applicability have been discussed earlier.^[3] We note that for the calculation of the π^-p interaction at 7 Bev, the method requires knowledge of the $\pi\pi$ interaction cross section $\sigma_{\pi\pi}$, the dependence of this cross section on the pion energy (in the c.m.s. of the pions, hereafter referred to as the $\pi\pi$ system), and on their isotopic spin (here the three possible states are $T = 0, 1, 2$).

One of the authors^[2] has shown that the best description is probably given by the variant in which all three isostates of the $\pi\pi$ system occur with equal probability. Using this variant, we can readily write the cross section of the π^-p interaction in the form

$$\sigma_{\pi p} = \frac{1}{4\pi^3 m^2 E_L^2} \int dz \int dy V(z^2 - m^2 \mu^2) (y^2 - \mu^4) \left[\frac{1}{\kappa^2 + \mu^2} - \frac{1}{\kappa^2 + \mu^2 + 4p_0 P_1} \right] \sigma_{\pi\pi}(y) \{2\sigma_{3/2}(z) + \sigma_{1/2}(z)\}, \quad (1)$$

where $z = 1/2 (\mathfrak{M}_1^2 - m^2 - \mu^2)$, $y = 1/2 (\mathfrak{M}_2^2 - 2\mu^2)$,

*In^[2] one of the authors considered mainly the isotopic relations in the $\pi\pi$ interaction.

FIG. 1. One-meson diagram of inelastic πp interactions.



$\mathfrak{M}_2$ is the pion energy in the $\pi\pi$ system; $\mathfrak{M}_1$ is the total energy of the pion and nucleon emitted from node 1 (Fig. 1) in their c.m.s.; E_0 and p_0 are the c.m.s. energy and momentum of the nucleon prior to the interaction; E_1 and P_1 are the total energy and momentum of the particles emitted from node 1 in the c.m.s. of the process; E_L is the pion energy in the laboratory system (l.s.); and $\kappa^2 = 2(E_0 E_1 - p_0 P_1) - \mathfrak{M}_1^2 - m^2$. The number of pions emitted from node 2 is even. In other respects, the number of pions (at both nodes 1 and 2) is arbitrary. The total number is limited by the conservation laws.

To see how well this picture agrees with the experimental data,^[1,4] we calculated the transverse momentum distribution, the c.m.s. angular distribution, and the l.s. energy distribution of the nucleons. We note that it is necessary to know for the calculation of these characteristics the value of $\sigma_{\pi\pi}(\mathfrak{M}_2)$ in the interval $0.3 < \mathfrak{M}_2 < 2$ Bev. The details of the $\pi\pi$ interaction (multiplicity in inelastic $\pi\pi$ interactions, angular and energy distributions of the pions in the $\pi\pi$ system) do not affect the results of this calculation.

We carried out the calculations under the following limiting assumptions: a) the quantity $\sigma_{\pi\pi}(\mathfrak{M}_2)$ depends weakly on the energy $\mathfrak{M}_2$, it can be assumed constant and equal to $\bar{\sigma}_{\pi\pi}$; b) the quantity $\sigma_{\pi\pi}(\mathfrak{M}_2)$ has a sharp maximum at $\mathfrak{M}_2 \sim 0.4 - 1$ Bev. It is equal to $\sigma_{\pi\pi}^{(r)}$ inside this region and is very small outside it. (Fraser and

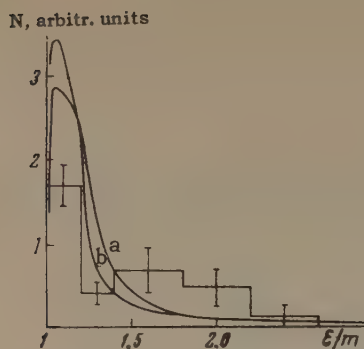


FIG. 2. Energy distribution of the nucleons in the l.s. Curves a and b are calculated by variants a and b, respectively; ε is the l.s. energy of the nucleon. The curves and histogram are normalized to the same area. The histogram represents the experimental data.

Fulco^[5] have discussed the possibility that such a "resonance" occurs.) We note that the characteristic values of the virtualness k^2 (k is the 4-momentum of the intermediate meson), as in ^[1], are $k^2 \sim (6\mu - 7\mu)^2$.

The total cross section of the inelastic πp interaction turned out to be $\sigma_{\pi p} = 0.3 \bar{\sigma}_{\pi\pi}$ in variant a and $\sigma_{\pi p} = 0.12 \sigma_{\pi\pi}^{(r)}$ in variant b. The experimental value of the inelastic cross section $\sigma_{\pi p}$ has been estimated earlier.^[3] (It should, of course, be borne in mind that this is a very preliminary estimate.) The estimate gave $\sigma_{\pi p} \approx 20$ mb. It can therefore be concluded that either $\bar{\sigma}_{\pi\pi} \approx 66$ mb in variant a or $\sigma_{\pi\pi}^{(r)} \approx 160$ mb in variant b. These values do not contradict other very approximate estimates.^[6,7]

The l.s. energy distribution of the nucleons, their c.m.s. angular distribution, and the transverse momentum distribution (in arbitrary units) are shown in Figs. 2, 3, and 4, respectively.

The angular distribution and the $p_{\perp}$ distribution calculated by variants a and b are practically the same. The energy distributions of the nucleons

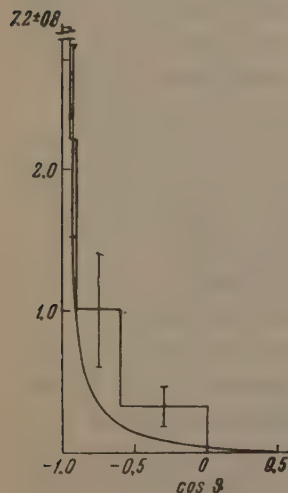
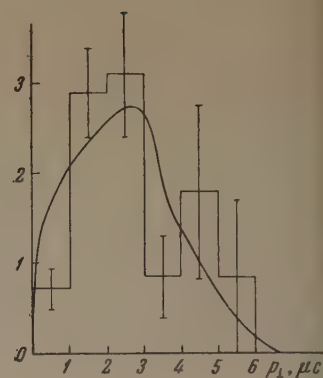


FIG. 3. Angular distribution of the nucleons in the c.m.s.

FIG. 4. Transverse momentum distribution of the nucleons in the c.m.s.



differ little. Hence from a comparison of these characteristics with experiment we can draw no conclusions as to the dependence of $\sigma_{\pi\pi}$ on the energy $\mathfrak{M}_2$.

The experimental data are represented in the figures by the histograms. Comparing these data, we see that: 1) the transverse momentum distribution is in satisfactory agreement with the experimental data; 2) the calculated energy distribution of the recoil nucleon and the angular distribution are in somewhat poorer agreement with the experimental data.

In the experimental distributions, cases with larger angles of the nucleons (in the c.m.s.) and high recoil-nucleon energies (in the l.s.) have greater weight.* Possibly, this is due to the insufficient accuracy of the experiment and will change when the accuracy is improved. On the other hand, it is possible that the entire process does not take place here in accordance with the foregoing scheme and that there is a mixture of cases occurring through other channels.

It is possible, for example, that part of the cases correspond to diagrams in which the nucleon is not excited and its interaction with the pion is proportional to $g(\psi\gamma_5\psi)\varphi$. (This scheme has been suggested by Chew and Low^[8] and has been considered several times^[6,7] at lower energies — $E_L \approx 1 - 3$ Bev.) Calculation shows[†] that in this case the angular and energy distributions of the nucleons are quite broad. However, this calculation cannot be made at 7 Bev, since the values of the virtualness k^2 and of the quantity $\mathfrak{M}_2$ are appreciably larger here than in the process pictured in Fig. 1. In fact, the values of k^2 which contribute significantly to the Chew-Low process are of the

*We note that the experimentally found angular and energy distributions of the nucleons in the case of the πp interaction at 7 Bev is appreciably broader than the similar characteristics of NN interactions at 9 Bev.^[3]

[†]We do not reproduce this calculation here, since it has been given in detail elsewhere.^[6-8]

order $k^2 \approx (13\mu)^2$ and $\mathfrak{M}_2 \approx 2 - 3$ Bev. It was shown earlier^[9] that restrictions on the virtualness must be introduced for such values of k^2 and $\mathfrak{M}_2$. Since the restricting procedure is not unique, this process cannot, at present, be calculated correctly.

Consequently, we did not analyze this process in detail in the present study and considered only the process of Fig. 1.

We return to the question of the dependence of $\sigma_{\pi\pi}$ on the energy $\mathfrak{M}_2$. As we have already mentioned, the observed characteristics of the recoil nucleons are not sensitive to this energy and do not provide the required information. In order to find the value of $\sigma_{\pi\pi}(\mathfrak{M}_2)$ in the region $\mathfrak{M}_2 > 1$ Bev, it is necessary to consider in greater detail the characteristics not of the nucleons, but of the secondary pions in the given process.

It can be expected that for $\mathfrak{M}_2 < 1.2$ Bev the $\pi\pi$ interaction will be primarily elastic. In fact, the inelastic interaction will be important only if all the secondary pions in the $\pi\pi$ system are relativistic, i.e., if they have an energy $\epsilon \gtrsim 2\mu$.

The minimum number of secondary particles in an inelastic collision is four. Hence it can be assumed that the interaction is elastic up to the energy $\mathfrak{M}_2 \approx 1.2$ Bev; for $\mathfrak{M}_2 > 1.2$ Bev the inelastic part will be important; in this case the elastic part will have a diffractive character.

The cross section of the inelastic $\pi\pi$ interaction $\sigma_{\pi p}^{(i)}$ due to the inelastic $\pi\pi$ interaction in the process under consideration (Fig. 1) and calculated according to formula (1) turns out to be $\sigma_{\pi p}^{(i)} = 0.15 \sigma_{\pi\pi}^{(i)}$, where $\sigma_{\pi\pi}^{(i)}$ is the inelastic $\pi\pi$ cross section averaged over the energy interval $1.2 < \mathfrak{M}_2 < 2$ Bev. To separate such interactions, we can select cases in which more than two secondary pions are emitted forward in the c.m.s.

Of basic interest here is the elastic (diffractive) $\pi\pi$ interaction (resulting from the existence of the inelastic $\pi\pi$ interaction). It leads to specific characteristics of the experimentally observed phenomena which allow it to be separated quite distinctly.

First, the sign of the diffraction-scattered pion should be negative.

Second, for $\mathfrak{M}_2 > 1.2$ Bev the l.s. energy of the fastest diffraction-scattered pion is $\epsilon_L \gtrsim 5$ Bev, according to the calculations with formula (1). In interactions of another type, the energy of the fastest pions is appreciably less. For example, in an elastic, but nondiffractive interaction (for $\mathfrak{M}_2 < 1.2$ Bev), $\epsilon_L \approx 3.8$ Bev; for an inelastic $\pi\pi$ interaction (with $\mathfrak{M}_2 > 1.2$ Bev) $\epsilon_L \approx 2$ Bev.

Third, the angle at which the diffraction-scattered pions of l.s. energy $\epsilon_L \sim 5$ Bev are emitted is small. It can be estimated in the following way. The basic contribution to the transverse component of these pions comes from the quantity $k_\perp$ (transverse momentum of the intermediate virtual pions). The $k_\perp$ distribution was calculated on the basis of expression (1). The curve of $d\sigma/dk_\perp$ has a maximum at $k_\perp = 2\mu$ and reaches half the maximum value at $k_\perp = 0.5\mu$ and $k_\perp = 4.5\mu$. Therefore the l.s. angles of the diffraction-scattered pions will lie within the limits $0.014 < \vartheta_L < 0.13$ ($1^\circ < \vartheta_L < 9^\circ$).

Using formula (1) and taking $\sigma_{\pi\pi}^{(d)}$ to be constant, we find the πp interaction cross section due to the diffractive $\pi\pi$ interaction:

$$\sigma_{\pi p}^{(d)} = 0.15 \sigma_{\pi\pi}^{(d)}. \quad (2)$$

Summing up, we can state that the diffractive interaction between the incident pion and a virtual pion can be separated experimentally if we select cases in which a negative pion is emitted in the forward direction with an energy greater than 5 Bev at an angle less than 9° l.s.

According to the experimental data,^[4] the fraction of such cases in πp interactions at 7 Bev is 0.15, i.e., the cross section of this process is $\sigma_{\pi p}^{(d)} \approx 3$ mb. We can thus conclude that the elastic diffraction-scattering $\pi\pi$ cross section is, according to (2), $\sigma_{\pi\pi}^{(d)} \approx 20$ mb in the energy region $1.2 < \mathfrak{M}_2 < 2$ Bev. We note that according to^[4] the greater part of the pions of energy $\epsilon_L \gtrsim 5$ Bev proved to be negative. This indicates that the elastic $\pi\pi$ scattering in this energy region occurs in most cases without the pion experiencing charge exchange, which is in agreement with the proposed picture of the interaction.

In conclusion, the authors express their gratitude to E. L. Feinberg for fruitful discussions and to the authors of^[1,4] for making their data available to us and for discussions.

¹ Belyakov, Wang, Glagolev, Dalkhazhav, Lebedev, Mel'nikova, Nikitin, Petrzilka, Sviridov, Suk, and Tolstov, JETP 39, 937 (1960), Soviet Phys. JETP 12, 650 (1961).

² I. M. Dremin, JETP 39, 130 (1960), Soviet Phys. JETP 12, 94 (1961).

³ Gramenitskii, Dremin, Maksimenko, and Chernavskii, JETP 40, 1093 (1961), Soviet Phys. JETP 13, 771 (1961).

⁴ Birger, Wang, Wang, Ting, Katicheva, Kladnitskaya, Kopylova, Lyubimov, Nguyen, Nikitin,

Podgoretskii, Smorodin, Solov'ev, and Trka, JETP 41, 1461 (1961), Soviet Phys. JETP 14, in press.

⁵W. R. Fraser and J. R. Fulco, Phys. Rev. Lett. 2, 365 (1959).

⁶Anderson, Burke, Carmony, and Schmitz, Proc. Intern. Conf. on High Energy Physics, Rochester, 1960, p. 58.

⁷Bonsignori, Bortolani, and Stanghellini, Nuovo cimento 18, 1237 (1960).

⁸G. F. Chew and F. E. Low, Phys. Rev. 113, 1640 (1959).

⁹I. M. Dremin and D. S. Chernavskii, JETP 40, 1333 (1961), Soviet Phys. JETP 13, 938 (1961).

Translated by E. Marquit
148

HIGH-FREQUENCY DIELECTRIC CONSTANT OF A PLASMA

V. P. SILIN

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 861-870 (September, 1961)

An expression is found for the complex dielectric constant of a dilute totally ionized plasma; the expression is accurate to quadratic terms in the number of particles per unit volume and applies for frequencies much higher than the electron Langmuir frequency. An isotropic plasma and a plasma in a magnetic field are considered. In the latter case the applied frequency is assumed to be greater than the Langmuir frequency, but can be greater or smaller than the gyromagnetic frequencies of the plasma particles.

1. The present communication is concerned with the high-frequency dielectric constant of a plasma. Specifically, we are interested in the frequency region in which the applied frequency ω is much higher than the electron Langmuir frequency $\omega_{Le} = \sqrt{4\pi e^2 N_e / m}$. On the other hand, we shall assume that the frequency is much lower than the frequency $\omega_{max} = (\kappa T)^{3/2} (2m)^{-1/2} / |e|e$. To describe a plasma under these conditions we find it convenient to use the kinetic equation for rapid processes given earlier by the author.^[1]

Under the conditions given above it is well-known^[2] that the imaginary part of the dielectric constant of an isotropic plasma must be modified; the correction is quadratic in the number of particles per unit volume and exhibits a frequency dependence of the form $\omega^{-3} \ln(\omega/\omega_{max})$. Below we obtain the corresponding real correction, which exhibits a frequency dependence of the form $\omega^{-3} \times \text{sign } \omega$. Finally, we obtain the correction to the dielectric tensor of a plasma in a strong magnetic field when neither the applied frequency nor the electron (ion) gyromagnetic frequency are small compared with the electron Langmuir frequency.

2. We find the dielectric constant of the plasma through the use of the kinetic equation for rapid steady-state processes. For a plasma in a spatially uniform alternating electric field $\mathbf{E}$ and a fixed magnetic field $\mathbf{B}$ this equation is:^[1]

$$\begin{aligned} \frac{\partial f_\alpha}{\partial t} + e_\alpha \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_\alpha \mathbf{B}] \right) \frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha} = \sum_\beta N_\beta \frac{\partial}{\partial \mathbf{p}_\alpha} \int d\mathbf{p}_\beta d\mathbf{r}_\beta \\ \times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial \mathbf{r}_\alpha^i} \int_{-\infty}^0 d\tau \left\{ \frac{\partial}{\partial \mathbf{r}_\alpha^i} U_{\alpha\beta}(|\mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha] \right. \\ \left. - \mathbf{R}_\beta[t+\tau, t, \mathbf{p}_\beta, \mathbf{r}_\beta]) \right\} \left\{ \frac{\partial}{\partial P_\alpha^j[t+\tau, t, \mathbf{p}_\alpha]} \right. \\ \left. - \frac{\partial}{\partial P_\beta^j[t+\tau, t, \mathbf{p}_\beta]} \right\} f_\alpha(\mathbf{P}_\alpha'[t+\tau, t, \mathbf{p}_\alpha], \mathbf{R}_\alpha[t \\ + \tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha], t + \tau) f_\beta(\mathbf{P}_\beta[t+\tau, t, \mathbf{p}_\beta], \mathbf{R}_\beta[t \\ + \tau, t, \mathbf{p}_\beta, \mathbf{r}_\beta], t + \tau). \end{aligned} \quad (1)^*$$

$$*[\mathbf{v}_\alpha \mathbf{B}] = \mathbf{v}_\alpha \times \mathbf{B}; (\mathbf{B} \mathbf{p}_\alpha) = \mathbf{B} \cdot \mathbf{p}_\alpha.$$

Here

$$\begin{aligned} \mathbf{P}_\alpha[t+\tau, t, \mathbf{p}_\alpha] = \mathbf{B} \frac{(\mathbf{B} \mathbf{p}_\alpha)}{B^2} - \sin \Omega_\alpha \tau \frac{[\mathbf{B} \mathbf{p}_\alpha]}{B} - \cos \Omega_\alpha \tau \frac{[\mathbf{B} [\mathbf{B} \mathbf{p}_\alpha]]}{B^2} \\ + e_\alpha \int_t^{t+\tau} dt' \left\{ \mathbf{B} \frac{(\mathbf{B} \mathbf{E}(t'))}{B^2} - \frac{[\mathbf{B} \mathbf{E}(t')]}{B} \sin \Omega_\alpha (t + \tau - t') \right. \\ \left. - \frac{[\mathbf{B} (\mathbf{B} \mathbf{E}(t'))]}{B^2} \cos \Omega_\alpha (t + \tau - t') \right\}, \end{aligned} \quad (2)$$

$$\begin{aligned} \mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha] = \mathbf{r}_\alpha + \mathbf{B} \frac{(\mathbf{B} \mathbf{v}_\alpha)}{B^2} \tau - \frac{1 - \cos \Omega_\alpha \tau}{\Omega_\alpha} \frac{[\mathbf{B} \mathbf{v}_\alpha]}{B} \\ - \frac{\sin \Omega_\alpha \tau}{\Omega_\alpha} \frac{[\mathbf{B} [\mathbf{B} \mathbf{v}_\alpha]]}{B^2} + \frac{e_\alpha}{m_\alpha} \int_t^{t+\tau} dt' \int_{t'}^{t'+\tau} dt'' \left\{ \mathbf{B} \frac{(\mathbf{B} \mathbf{E}(t''))}{B^2} \right. \\ \left. - \frac{[\mathbf{B} \mathbf{E}(t'')]}{B} \sin \Omega_\alpha (t' - t'') - \frac{[\mathbf{B} (\mathbf{B} \mathbf{E}(t''))]}{B^2} \cos \Omega_\alpha (t' - t'') \right\}, \end{aligned} \quad (3)$$

where e_α , m_α , $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$, and $\mathbf{p}_\alpha$ are respectively the charge, mass, coordinate, velocity, and momentum of a particle of type α ; $\Omega_\alpha = e_\alpha B / m_\alpha c$ is the gyromagnetic frequency; N_α is the number of particles of type α per unit volume and, $U_{\alpha\beta}(\mathbf{r}) = e_\alpha e_\beta / r$.

Equation (1) has been obtained under the assumption of a weak particle interaction and does not apply for small impact parameters. In this connection, in integrating over the impact parameters below, we introduce a cutoff at ρ_{min} . On the other hand we have not considered shielding of the Coulomb interaction at large distances in Eq. (1). Equation (1) may not be applicable to an analysis of collisions at high impact parameters in sufficiently slow processes, in which case we must introduce a cutoff at ρ_{max} .

3. We first consider an isotropic plasma with no fixed magnetic field. We neglect spatial dis-

*If collisions are neglected the following relation holds:

$$f_\alpha(\mathbf{P}_\alpha[t+\tau, t, \mathbf{p}_\alpha], \mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha], t + \tau) = f_\alpha(\mathbf{p}_\alpha, \mathbf{r}_\alpha, t),$$

and allows us to transform the arguments of the functions in the right side of Eq. (1).

persion in the dielectric constant, assuming the particle distribution to be uniform in space. Assuming a small departure from the Maxwellian distribution $f_{\alpha}^{(0)}$ and linearizing the kinetic equation, taking account of the fact that δf_{α} is proportional to the electric field, which is assumed to be weak, we have

$$\begin{aligned} \frac{\partial \delta f_{\alpha}}{\partial t} - \frac{e_{\alpha}}{\kappa T} \mathbf{E} \mathbf{v}_{\alpha} f_{\alpha}^{(0)} &= \sum_{\beta} N_{\beta} \frac{\partial}{\partial p_{\beta}^i} \int d\mathbf{p}_{\beta} d\mathbf{r}_{\beta} \\ &\times \int_{-\infty}^0 d\tau \left\{ \frac{\partial U_{\alpha\beta}(|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}| + (\mathbf{v}_{\alpha} - \mathbf{v}_{\beta})\tau)}{\partial r_{\alpha}^i} \right. \\ &\times \frac{\partial U_{\alpha\beta}(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta})}{\partial r_{\alpha}^i} \left[\frac{\partial}{\partial p_{\alpha}^i} - \frac{\partial}{\partial p_{\beta}^i} \right] [f_{\alpha}^{(0)} \delta f_{\beta}(\mathbf{p}_{\beta}, t + \tau) \\ &+ \delta f_{\alpha}(\mathbf{p}_{\alpha}, t + \tau) f_{\beta}^{(0)}] - \frac{f_{\alpha}^{(0)} f_{\beta}^{(0)}}{(\kappa T)^2} \frac{\partial U_{\alpha\beta}(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta})}{\partial r_{\alpha}^i} U_{\alpha\beta}(|\mathbf{r}_{\alpha} \\ &- \mathbf{r}_{\beta}| + (\mathbf{v}_{\alpha} - \mathbf{v}_{\beta})\tau) (e_{\alpha} \mathbf{v}_{\alpha} + e_{\beta} \mathbf{v}_{\beta}, \mathbf{E}(t + \tau)) \Big\}. \end{aligned} \quad (4)$$

In solving Eq. (4) we assume that the collision integral is small. If there is a periodic time dependence ($e^{-i\omega t}$) the inequality $\omega \gg \nu_{\text{eff}}$ must be satisfied, where ν_{eff} is defined below. In the first approximation

$$\delta f_{\alpha}^{(1)} = i \frac{e_{\alpha}}{\kappa T} \frac{\mathbf{E} \mathbf{v}_{\alpha}}{\omega} f_{\alpha}^{(0)}. \quad (5)$$

Substituting Eq. (5) for δf in Eq. (4), we have as a second approximation

$$\begin{aligned} \delta f_{\alpha}^{(2)} &= \frac{1}{\omega^2} \sum_{\beta} N_{\beta} \frac{\partial}{\partial p_{\beta}^i} \int d\mathbf{p}_{\beta} d\mathbf{r}_{\beta} \frac{\partial U_{\alpha\beta}(|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|)}{\partial r_{\alpha}^i} \int_{-\infty}^0 d\tau e^{-i\omega\tau} \\ &\times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}| + (\mathbf{v}_{\alpha} - \mathbf{v}_{\beta})\tau)}{\partial r_{\alpha}^i} \frac{1}{(\kappa T)^2} f_{\alpha}^{(0)} f_{\beta}^{(0)} \left(\frac{e_{\beta}}{m_{\beta}} - \frac{e_{\alpha}}{m_{\alpha}} \right) E_j. \end{aligned} \quad (6)$$

An expression for the current density can be obtained from Eqs. (5) and (6)

$$\mathbf{j} = \sum_{\alpha} e_{\alpha} N_{\alpha} \int d\mathbf{p}_{\alpha} \mathbf{v}_{\alpha} \delta f_{\alpha}, \quad (7)$$

and can then be used to find the complex conductivity tensor σ_{ij} ($j_i = \sigma_{ij} E_j$) or the complex dielectric tensor $\epsilon_{ij} = \delta_{ij} + 4\pi i \sigma_{ij} / \omega$. For the isotropic plasma being considered here these tensors are diagonal and, from Eqs. (5) - (7), we have

$$\begin{aligned} \epsilon(\omega) &= 1 - \sum_{\alpha} \frac{4\pi e_{\alpha}^2 N_{\alpha}}{\omega^2 m_{\alpha}} + \frac{4\pi i}{\omega^3} \sum_{\alpha\beta} \frac{e_{\alpha}}{m_{\alpha}} \left(\frac{e_{\alpha}}{m_{\alpha}} - \frac{e_{\beta}}{m_{\beta}} \right) \frac{N_{\alpha} N_{\beta}}{\kappa T} \int d\mathbf{p}_{\alpha} d\mathbf{p}_{\beta} \\ &\times f_{\alpha}^{(0)} f_{\beta}^{(0)} \int_{-\infty}^0 d\tau e^{-i\omega\tau} \int_{k_{\min}}^{k_{\max}} \frac{dk}{(2\pi)^3} \frac{(4\pi e_{\alpha} e_{\beta})^2}{3k^2} \exp\{ik(\mathbf{v}_{\alpha} - \mathbf{v}_{\beta})\tau\}, \end{aligned} \quad (8)$$

where $k_{\max} = \rho_{\min}^{-1} = \kappa T / |e_{\alpha} e_{\beta}|$ while $k_{\min} = \rho_{\max}^{-1} \approx r_D^{-1}$ (r_D is the Debye radius).

If terms containing positive powers of the electron-ion mass ratio are neglected in the right side

of Eq. (8) and if it is assumed that only one type of ion is present, we have

$$\epsilon(\omega) = 1 - \frac{\omega_{Le}^2}{\omega^2} + i \frac{\omega_{Le}^2}{\omega^3} \frac{4}{3} \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{3/2}} F(\omega); \quad (9)$$

$$F(\omega) = \int_0^{\infty} \frac{d\tau}{\tau} e^{i\omega\tau} \left[\Phi\left(\tau \sqrt{\frac{\kappa T}{2m}} k_{\max}\right) - \Phi\left(\tau \sqrt{\frac{\kappa T}{2m}} k_{\min}\right) \right],$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (10)$$

Keeping in mind that $k_{\max} \gg k_{\min}$ and also that these quantities are determined to an accuracy of order unity, we can write Eq. (10) in the form

$$\begin{aligned} F(\omega) &= F'(\omega) + iF''(\omega) = \int_{\tau_{\min}}^{\tau_{\max}} \frac{d\tau}{\tau} \cos \omega\tau + i \int_{\tau_{\min}}^{\tau_{\max}} \frac{d\tau}{\tau} \sin \omega\tau, \\ \tau_{\min} &= \sqrt{2m/\kappa T} k_{\max}^{-1}, \quad \tau_{\max} = \sqrt{2m/\kappa T} k_{\min}^{-1} \approx \omega_{Le}^{-1}. \end{aligned} \quad (11)$$

When $\omega \ll \omega_{Le}$,

$$F'(\omega) = \ln \frac{k_{\max}}{k_{\min}}, \quad F''(\omega) \approx \frac{\omega}{\omega_{Le}}. \quad (12)$$

The last expression for F'' leads to a small correction (proportional to ω^{-2}) in $\epsilon(\omega)$. Hence, ω_{Le}^2 changes by an amount $\Delta \omega_{Le}^2$, where

$$\Delta \approx \frac{4}{3} \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{3/2}} \frac{1}{\omega_{Le}}.$$

Equation (12) for $F'(\omega)$ leads to the usual effective collision frequency, which then yields the following expression for the dielectric constant:^[2-4]

$$\epsilon(\omega) = 1 - \frac{\omega_{Le}^2}{\omega^2} (1 + \Delta) + i \frac{\omega_{Le}^2}{\omega^3} \nu_{\text{eff}}^{(0)}, \quad (13)$$

where

$$\nu_{\text{eff}}^{(0)} = \frac{4}{3} \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{3/2}} \ln \left(\frac{\kappa T}{|e e_i|} r_D \right). \quad (14)$$

At frequencies much greater than the electron Langmuir frequency ($\omega \gg \omega_{Le}$) we have from Eq. (11)

$$F'(\omega) = \ln \left(\frac{k_{\max}}{\gamma |\omega|} \sqrt{\frac{\kappa T}{2m}} \right), \quad F''(\omega) = \frac{\pi}{2} \text{sign } \omega; \quad (15)$$

$\gamma = 1.781$ is the Euler constant. Substituting (15) in (9) we have

$$\epsilon(\omega) = \epsilon' + i\epsilon'' = 1 - \frac{\omega_{Le}^2}{\omega^2} + i \frac{\omega_{Le}^2}{\omega^3} \nu_{\text{eff}}^{(\omega)} - \frac{\omega_{Le}^2}{\omega^3} \omega_{\text{eff}} \text{sign } \omega, \quad (16)$$

where

$$\omega_{\text{eff}} = \frac{(2\pi)^{3/2} (ee_i)^2 N_i}{3 \sqrt{m} (\kappa T)^{3/2}}, \quad (17)$$

while $\nu_{\text{eff}}^{(\omega)}$ is of the form known from the theory of absorption of radio waves in interstellar gases:^[2]

$$\nu_{\text{eff}}^{(\omega)} = \frac{4}{3} \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{3/2}} \ln \left| \frac{(\kappa T)^{1/2}}{\gamma \omega \sqrt{2m} |ee_i|} \right|. \quad (18)$$

The quantity ν_{eff} differs from ω_{eff} by the large logarithmic term. Since the logarithmic term depends on the choice of k_{max} , there is some question as to the usefulness of keeping the term proportional to ω_{eff} . However, ν_{eff} appears in the imaginary part of the dielectric constant while ω_{eff} appears in the real part, so that we are justified in keeping the term proportional to ω_{eff} .

There is an important difference in the correction to the real part of $\epsilon(\omega)$ when $\omega \gg \omega_{\text{Le}}$ and $\omega \ll \omega_{\text{Le}}$. The absolute magnitude of the correction is smaller in the second case than in the first (low frequency); however, a new dependence on frequency arises in the region $\omega \gg \omega_{\text{Le}}$ and in principle, makes it possible to observe the corresponding correction. The difficulty of an observation of this kind is due to the necessity for satisfying the condition $\omega \ll (\kappa T)^{3/2} m^{-1/2} / |ee_i|$. For example, in the expression for the refractive index

$$n = \sqrt{\epsilon'} \approx 1 - \frac{\omega_{\text{Le}}^2}{2\omega^2} - \frac{1}{8} \frac{\omega_{\text{Le}}^4}{\omega^4} - \frac{1}{2} \frac{\omega_{\text{Le}}^2 \omega_{\text{eff}}}{\omega^3} - \frac{1}{16} \frac{\omega_{\text{Le}}^6}{\omega^6} + \dots \quad (19)$$

the third term ($\sim \omega^{-4}$) is always greater than the fourth, which is proportional to ω^{-3} . Thus the correction will appear only when the frequency dependence of the refractive index is determined with very high accuracy. We note that the fourth term of the right side of Eq. (19) is smaller than the third, but greater than the fifth, if the following condition holds:

$$N_e^{1/2} (\kappa T)^{1/2} m^{-1/2} \ll \omega \ll (\kappa T)^{1/2} m^{-1/2} / |ee_i|.$$

This frequency range is rather wide since it is given by the relation $|ee_i| N_e^{1/3} \ll \kappa T$.

4. We now consider a plasma in a fixed magnetic field. In this case the linearized kinetic equation for weak departures from the Maxwellian distribution can be written in the form

$$\begin{aligned} \frac{\partial \delta f_\alpha}{\partial t} + \frac{e_\alpha}{c} [\mathbf{v}_\alpha \mathbf{B}] \frac{\partial \delta f_\alpha}{\partial \mathbf{p}_\alpha} - \frac{e_\alpha}{\kappa T} \mathbf{E} \mathbf{v}_\alpha f_\alpha^{(0)} = \sum_\beta N_\beta \frac{\partial}{\partial \mathbf{p}_\alpha} \int d\mathbf{p}_\beta d\mathbf{r}_\beta \\ \times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r_\alpha^i} \int_{-\infty}^0 d\tau \left\{ \left[\frac{\partial}{\partial r_\alpha^i} U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{p}_\alpha, \mathbf{r}_\alpha) \right. \right. \\ \left. \left. - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta) \right) \right] \left[\frac{\partial}{\partial P_\alpha^{0j}(\tau, \mathbf{p}_\alpha)} - \frac{\partial}{\partial P_\beta^{0j}(\tau, \mathbf{p}_\beta)} \right] [f_\alpha^{(0)} \delta f_\beta \\ \times (\mathbf{P}_\beta^0(\tau, \mathbf{p}_\beta), t + \tau) + f_\beta^{(0)} \delta f_\alpha(\mathbf{P}_\alpha^{(0)}(\tau, \mathbf{p}_\alpha), t + \tau)] \\ - U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{p}_\alpha, \mathbf{r}_\alpha) - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|) \\ \times \frac{1}{(\kappa T)^2} f_\alpha^{(0)} f_\beta^{(0)} \left(\mathbf{E}(t + \tau), \frac{e_\alpha}{m_\alpha} \mathbf{P}_\alpha^0(\tau, \mathbf{p}_\alpha) + \frac{e_\beta}{m_\beta} \mathbf{P}_\beta^0(\tau, \mathbf{p}_\beta) \right) \right\}. \quad (20) \end{aligned}$$

Here, $\mathbf{P}^0$ and $\mathbf{R}^0$ are given by Eqs. (2) and (3) if the electric field is set equal to zero.

When the field is periodic in time ($e^{-i\omega t}$) Eq. (20) can be solved by perturbation methods by taking the right side to be small, so long as the condition $|\omega^2 \pm \Omega_\alpha^2| \gg \nu_{\text{eff}}^2$ is satisfied. We assume below that this condition is satisfied. Then, assuming that the collision integral can be neglected, we have

$$\delta f_\alpha^{(1)}(\mathbf{p}_\alpha, t) = \frac{e_\alpha f_\alpha^{(0)}}{m_\alpha \kappa T} A_{sr}(\omega, \Omega_\alpha) \rho_\alpha^r E_s e^{-i\omega t}, \quad (21)$$

$$A_{sr} = \left\{ \frac{B_s B_r}{B^2} \frac{i}{\omega} - \frac{\Omega_\alpha}{\omega^2 - \Omega_\alpha^2} e_{slr} \frac{B_l}{B} - \frac{i\omega}{\omega^2 - \Omega_\alpha^2} \frac{B_s B_r - B^2 \delta_{sr}}{B^2} \right\}. \quad (22)$$

Here, e_{slr} is a completely antisymmetric tensor. Using Eq. (21) we obtain the following equation for the second approximation correction to the non-equilibrium part of the distribution function:

$$\begin{aligned} \frac{\partial \delta f_\alpha^{(2)}}{\partial t} + \frac{e_\alpha}{c} [\mathbf{v}_\alpha \mathbf{B}] \frac{\partial \delta f_\alpha^{(2)}}{\partial \mathbf{p}_\alpha} \\ = \sum_\beta N_\beta \frac{\partial}{\partial \mathbf{p}_\alpha} \int d\mathbf{p}_\beta d\mathbf{r}_\beta \frac{f_\beta^{(0)} f_\alpha^{(0)}}{\kappa T} \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r_\alpha^i} \\ \times \int_{-\infty}^0 d\tau e^{-i\omega\tau} \left\{ \frac{\partial}{\partial r_\alpha^i} U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{p}_\alpha, \mathbf{r}_\alpha) - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|) \right\} \\ \times \left[\frac{e_\alpha}{m_\alpha} A_{sj}(\omega, \Omega_\alpha) - \frac{e_\beta}{m_\beta} A_{sj}(\omega, \Omega_\beta) \right] e^{-i\omega t} E_s. \quad (23) \end{aligned}$$

In the case being considered (steady-state periodic process) the solution of this equation can be written in the form

$$\begin{aligned} \delta f_\alpha^{(2)}(\mathbf{p}_\alpha, t) = \int_{-\infty}^t dt' \sum_\beta N_\beta \frac{\partial}{\partial \mathbf{p}_\alpha} \int d\mathbf{p}_\beta d\mathbf{r}_\beta \frac{f_\beta^{(0)} f_\alpha^{(0)}}{\kappa T} \\ \times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r_\alpha^i} \int_{-\infty}^0 d\tau e^{-i\omega\tau} \left\{ \frac{\partial}{\partial r_\alpha^i} U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{P}_\alpha^0[t' \right. \\ \left. - t, \mathbf{p}_\alpha], \mathbf{r}_\alpha) - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|) \right\} \left[\frac{e_\alpha}{m_\alpha} A_{sj}(\omega, \Omega_\alpha) \right. \\ \left. - \frac{e_\beta}{m_\beta} A_{sj}(\omega, \Omega_\beta) \right] e^{-i\omega t'} E_s. \quad (24) \end{aligned}$$

To find the complex dielectric tensor we substitute (21) and (24) in (7) and integrate over momentum and time (t'). In particular, we assume that

$$\int_0^\infty dt e^{i\omega t} \frac{\partial}{\partial \mathbf{p}_\alpha} P_\alpha^r(t, \mathbf{p}_\alpha) = A_{rt}(\omega, -\Omega_\alpha). \quad (25)$$

Thus

$$\begin{aligned} \epsilon_{ij}(\omega) = \delta_{ij} + \sum \frac{4\pi e_\alpha^2 N_\alpha}{\omega m_\alpha} i A_{ji}(\omega, \Omega_\alpha) - \frac{4\pi i}{\omega} \sum_{\alpha\beta} \frac{e_\alpha}{m_\alpha} A_{tr}(\omega, -\Omega_\alpha) \\ \times \left[\frac{e_\alpha}{m_\alpha} A_{js}(\omega, \Omega_\alpha) - \frac{e_\beta}{m_\beta} A_{js}(\omega, \Omega_\beta) \right] \frac{N_\alpha N_\beta}{\kappa T} (4\pi e_\alpha e_\beta)^2 \\ \times \int_{-\infty}^0 d\tau e^{-i\omega\tau} \int_{k_{\min}}^{k_{\max}} \frac{dk}{(2\pi)^3} \frac{k_r k_s}{k^4} \exp \left\{ -\frac{\kappa T}{2} \left[\frac{1}{m_\alpha} + \frac{1}{m_\beta} \right] \left(\frac{kB}{B} \right)^2 \tau^2 \right. \\ \left. - 2\kappa T \frac{B^2 k^2 - (Bk)^2}{B^2} \left[\frac{\sin^2(\Omega_\alpha \tau/2)}{m_\alpha \Omega_\alpha^2} + \frac{\sin^2(\Omega_\beta \tau/2)}{m_\beta \Omega_\beta^2} \right] \right\}. \quad (26) \end{aligned}$$

Assuming that the plasma contains electrons and only one ion species and neglecting corrections of the order of the electron-ion mass ratio, we have

$$\varepsilon_{ij}(\omega) = \varepsilon_{ij}^{(0, h)} + \delta\varepsilon_{ij}^{(a)} + \delta\varepsilon_{ij}^{(h)}, \quad (27)$$

where $\varepsilon_{ij}^{(0, h)}$ is the Hermitian part of the dielectric tensor, obtained if the collision integral is neglected completely

$$\varepsilon_{ij}^{(0, h)} = \delta_{ij} - \frac{\omega_{Le}^2}{\omega^2} \left\{ \frac{B_i B_j}{B^2} - \frac{\omega^2}{\Omega_e} \left[\frac{\Omega_e}{\omega^2 - \Omega_e^2} - \frac{\Omega_i}{\omega^2 - \Omega_i^2} \right] \frac{B_i B_j - \delta_{ij} B^2}{B^2} \right. \\ \left. + \frac{i\omega}{\Omega_e} \left[\frac{\Omega_e^2}{\omega^2 - \Omega_e^2} - \frac{\Omega_i^2}{\omega^2 - \Omega_i^2} \right] e_{jll} \frac{B_l}{B} \right\}, \quad (28)$$

while $\delta\varepsilon_{ij}^{(h)}$ and $\delta\varepsilon_{ij}^{(a)}$ are respectively the Hermitian and anti-Hermitian parts of the dielectric tensor, obtained when the collision integral is taken into account:

$$\delta\varepsilon_{ij} = \delta\varepsilon_{ij}^{(a)} + \delta\varepsilon_{ij}^{(h)} = i \frac{\omega_{Le}^2 \omega_{eff}}{\omega^3} \left\{ \frac{B_i B_j}{B^2} F_1(\omega) \right. \\ \left. + [F_1(\omega) + F_2(\omega)] T_{ij}^{\perp} \right\}, \quad (29)$$

where

$$T_{ij}^{\perp} = -\frac{2i\omega^3}{\Omega_e^2} e_{jll} \frac{B_l}{B} \left[\frac{\Omega_e}{\omega^2 - \Omega_e^2} - \frac{\Omega_i}{\omega^2 - \Omega_i^2} \right] \left[\frac{\Omega_e^2}{\omega^2 - \Omega_e^2} - \frac{\Omega_i^2}{\omega^2 - \Omega_i^2} \right] \\ - \frac{\omega^2}{\Omega_e^2} \frac{B_i B_j - \delta_{ij} B^2}{B^2} \left\{ \omega^2 \left[\frac{\Omega_e}{\omega^2 - \Omega_e^2} - \frac{\Omega_i}{\omega^2 - \Omega_i^2} \right]^2 \right. \\ \left. + \left[\frac{\Omega_e^2}{\omega^2 - \Omega_e^2} - \frac{\Omega_i^2}{\omega^2 - \Omega_i^2} \right]^2 \right\}.$$

The functions F_1 and F_2 are given by

$$F_1(\omega) = \frac{3}{\pi} \int_0^{\infty} d\tau e^{i\omega\tau} \int_{-1}^{+1} \frac{dx x^2}{\sqrt{\varphi(x, \tau)}} \left\{ \Phi(k_{max} \sqrt{\frac{\kappa T}{2m}} \sqrt{\varphi(x, \tau)}) \right. \\ \left. - \Phi(k_{min} \sqrt{\frac{\kappa T}{2m}} \sqrt{\varphi(x, \tau)}) \right\}, \quad (30)$$

$$F_2(\omega) = \frac{3}{2\pi} \int_0^{\infty} d\tau e^{i\omega\tau} \int_{-1}^{+1} dx (1 - 3x^2) \frac{1}{\sqrt{\varphi(x, \tau)}} \\ \times \left\{ \Phi(k_{max} \sqrt{\frac{\kappa T}{2m}} \sqrt{\varphi(x, \tau)}) \right. \\ \left. - \Phi(k_{min} \sqrt{\frac{\kappa T}{2m}} \sqrt{\varphi(x, \tau)}) \right\}, \quad (31)$$

where

$$\varphi(x, \tau) = \left(1 + \frac{m}{m_i} \right) x^2 \tau^2 + 4(1 - x^2) \left[\frac{\sin^2(\Omega_e \tau / 2)}{\Omega_e^2} \right. \\ \left. + \frac{m}{m_i} \frac{\sin^2(\Omega_i \tau / 2)}{\Omega_i^2} \right]. \quad (32)$$

The real parts of the functions F_1 and F_2 make contributions to the anti-Hermitian part of the dielectric tensor while the imaginary parts appear in the Hermitian part.

At frequencies much higher than the electron gyromagnetic frequency ($\omega \gg \Omega_e$)

$$F_1(\omega) \approx 2\pi^{-1} F(\omega), \quad F_2(\omega) \approx 0. \quad (33)$$

Thus, in this frequency region and when $\omega \gg \omega_{Le}$, we have

$$\delta\varepsilon_{ij}(\omega) = \frac{\omega_{Le}^2}{\omega^3} [i\nu_{eff}^{(\omega)} - \omega_{eff}] \left\{ \frac{B_i B_j}{B^2} + T_{ij}^{\perp} \right\}. \quad (34)$$

If ω and $\Omega_e \ll \omega_{Le}$, then $\delta\varepsilon_{ij}$ is of the same form but $\nu_{eff}^{(\omega)}$ and ω_{eff} must be replaced by $\nu_{eff}^{(0)}$ and $\Delta\omega_{Le}$ respectively. We assume below that the applied frequency is greater than the electron Langmuir frequency. In particular, this assumption allows us to write $k_{min} = 0$ in Eqs. (30) and (31).

We first analyze the Hermitian part of the correction to the dielectric tensor, for which purpose we must consider the imaginary parts of the functions F_1 and F_2 . Because the integrands of the corresponding imaginary parts do not have singularities at small τ we can set k_{max} equal to infinity in the appropriate formulas. Taking $\Omega_e < 0$, we have

$$F_1''(\omega) = -\frac{3}{\pi} \int_0^{\infty} \frac{d\xi}{\xi} \sin\left(\xi \frac{2\omega}{\Omega_e}\right) \left\{ \frac{1}{1 - \psi(\xi)} \right. \\ \left. - \frac{\psi(\xi)}{[1 - \psi(\xi)]^{3/2}} \ln \frac{1 + \sqrt{1 - \psi(\xi)}}{\sqrt{\psi(\xi)}} \right\}, \quad (35)$$

$$F_2''(\omega) = -\frac{3}{\pi} \int_0^{\infty} \frac{d\xi}{\xi} \sin(\xi 2\omega / \Omega_e) \left\{ -\frac{3}{2} \frac{1}{\sqrt{1 - \psi(\xi)}} \right. \\ \left. + \left[1 + \frac{3}{2} \frac{\psi(\xi)}{1 - \psi(\xi)} \right] \ln \frac{1 + \sqrt{1 - \psi(\xi)}}{\sqrt{\psi(\xi)}} \right\}, \quad (36)$$

where

$$\sin^2 \xi + \frac{e^2 m_i}{e_i^2 m} \sin^2 \left(\frac{e_i m}{m_i} \xi \right) \\ \psi(\xi) = \frac{[1 + m/m_i] \xi^2}{[1 + m/m_i] \xi^2}. \quad (37)$$

To obtain relatively simple expressions we consider several limiting cases assuming at all times, however, that the gyromagnetic electron frequency and the frequency ω are small compared with $\omega_{max} = (\kappa T)^{3/2} m^{-1/2} / |ee_i|$. At the highest frequency ω , the magnetic field has no effect on collisions; and the situation is given by Eqs. (33) and (34). Hence, in the following we assume that $\Omega_e \gg \omega$.

If this inequality is satisfied the basic contribution in the integrals in (35) and (36) comes from the region $\xi \gg 1$ in which $\psi(\xi)$ is small compared with unity. We must distinguish three regions of large values of ξ . First, the region $1 \ll \xi \ll \sqrt{m_i/m}$, where

$$\psi(\xi) \approx \xi^{-2} \sin^2 \xi. \quad (38)$$

Second, the region defined by the inequality

$$\sqrt{m_i/m} < \xi < |em_i/e_i m|,$$

where

$$\psi(\xi) \approx m/m_i. \quad (39)$$

Finally, the region $\xi > |em_i/eim|$, where

$$\psi(\xi) \approx \frac{1}{\xi^2} \frac{e^2}{e_i^2} \frac{m_i}{m} \sin^2\left(\frac{e_i}{e} \frac{m}{m_i} \xi\right), \quad (40)$$

Since $\psi(\xi)$ is small, we obtain immediately from Eq. (35)

$$F_1''(\omega) = \frac{3}{2} \text{sign } \omega. \quad (41)$$

The situation is somewhat more complicated for the function $F_2''(\omega)$. Here, corresponding to the three ranges of values of ξ and the corresponding values of the function $\psi(\xi)$, we have

$$F_2''(\omega) \approx \frac{3}{2} \ln \left| \frac{\Omega_e}{\omega} \right| \text{sign } \omega, \quad \Omega_e \sqrt{\frac{m}{m_i}} \ll \omega \ll \Omega_e, \quad (42)$$

$$F_2''(\omega) = \frac{3}{4} \left[\ln \frac{4m_i}{m} - 3 \right] \text{sign } \omega, \quad \Omega_i \ll \omega \ll \Omega_e \sqrt{\frac{m}{m_i}}, \quad (43)$$

$$F_2''(\omega) = \frac{3}{2} \ln \left| \frac{\Omega_i}{\omega} \right| \sqrt{\frac{m_i}{m}} \text{sign } \omega, \quad \omega \ll \Omega_i. \quad (44)$$

Equations (29), (41) – (44) allow us to write the Hermitian part of the correction to the complex dielectric tensor in the following form:

$$\delta \epsilon_{ij}^{(h)}(\omega) = -\frac{3}{2} \frac{\omega_{Le}^2 \omega_{eff}}{\omega^3} \text{sign } \omega \left\{ \frac{B_i B_j}{B^2} + \ln \left| \frac{\Omega_e}{\omega} \right| T_{ij}^{\perp} \right\}, \quad (45)$$

$$\Omega_e \sqrt{\frac{m}{m_i}} \ll \omega \ll \Omega_e;$$

$$\delta \epsilon_{ij}^{(h)}(\omega) = -\frac{3}{2} \frac{\omega_{Le}^2 \omega_{eff}}{\omega^3} \text{sign } \omega \left\{ \frac{B_i B_j}{B^2} + \frac{1}{2} \left[\ln \frac{4m_i}{m} - 1 \right] T_{ij}^{\perp} \right\}, \quad (46)$$

$$\Omega_i \ll \omega \ll \Omega_e \sqrt{\frac{m}{m_i}};$$

$$\delta \epsilon_{ij}^{(h)}(\omega) = -\frac{3}{2} \frac{\omega_{Le}^2 \omega_{eff}}{\omega^3} \text{sign } \omega \left\{ \frac{B_i B_j}{B^2} + \ln \left| \frac{\Omega_i}{\omega} \right| \sqrt{\frac{m_i}{m}} T_{ij}^{\perp} \right\}, \quad (47)$$

$$\omega \ll \Omega_i.$$

We now consider the anti-Hermitian correction. As in the analysis of the imaginary parts of F_1 and F_2 we can write $k_{\max} = \infty$ in the real part of F_2 . Then, $F_2'(\omega)$ assumes a form similar to (36) with the difference that the cosine appears in place of the sine. When $\omega \ll \Omega_e$, we have

$$F_2'(\omega) = \frac{3}{2\pi} \int_1^{\infty} \frac{d\xi}{\xi} \cos\left(\xi \frac{2\omega}{\Omega_e}\right) \left\{ \ln \frac{4}{\psi(\xi)} - 3 \right\}. \quad (48)$$

Corresponding to Eqs. (38) – (40), we have from Eq. (48)

$$F_2'(\omega) = \frac{3}{2\pi} \left[\ln \frac{\omega}{\Omega_e} \right]^2, \quad \Omega_e \sqrt{\frac{m}{m_i}} \ll \omega \ll \Omega_e; \quad (49)$$

$$F_2'(\omega) = \frac{3}{2\pi} \left\{ \ln \left| \frac{\Omega_e}{2\gamma\omega} \right| \left[\ln \frac{4m_i}{m} - 3 \right] - \frac{1}{4} \left[\ln \frac{m_i}{m} \right]^2 \right\},$$

$$\Omega_i \ll \omega \ll \Omega_e \sqrt{m/m_i}; \quad (50)$$

$$F_2'(\omega) = \frac{3}{2\pi} \left\{ \left(\ln \frac{\Omega_i}{\omega} \right)^2 + \ln \frac{m_i}{m} \ln \left| \frac{\Omega_i}{\omega} \right| + \frac{3}{4} \left[\ln \frac{m_i}{m} \right]^2 \right\},$$

$$\omega \ll \Omega_i. \quad (51)$$

We cannot write $k_{\max} = \infty$ in $F_1'(\omega)$. However, since the integrand in Eq. (30) is independent of magnetic field when $\tau \ll 1/\Omega_e$, an expression for $F_1'(\omega)$ can be written that applies for both $\omega \gg \Omega_e$ and $\omega \ll \Omega_e$:

$$F_1'(\omega) = \frac{2}{\pi} \int_{\xi_{min}}^1 \frac{d\xi}{\xi} \cos\left(\frac{2\omega}{\Omega_e} \xi\right) + \frac{3}{\pi} \int_1^{\infty} \frac{d\xi}{\xi} \cos\left(\frac{2\omega}{\Omega_e} \xi\right),$$

$$\xi_{min} = \sqrt{2m} \Omega_e |ee_i| (\kappa T)^{-1/2}. \quad (52)$$

In the case of immediate interest we have

$$F_1'(\omega) = \frac{2}{\pi} \ln \frac{(\kappa T)^{1/2}}{\gamma \sqrt{2m} \Omega_e |ee_i|} + \frac{3}{\pi} \ln \left| \frac{\Omega_e}{2\omega} \right|. \quad (53)$$

Then, using Eqs. (29), (49) – (51) and (53) we can write an expression for the anti-Hermitian part of the complex dielectric tensor

$$\delta \epsilon_{ij}^{(a)}(\omega) = i \left\{ \frac{B_i B_j}{B^2} [v_{eff}^{(\Omega)} + \delta v_{\parallel}(\omega)] + T_{ij}^{\perp} [v_{eff}^{(\Omega)} + \delta v_{\perp}(\omega)] \right\} \frac{\omega_{Le}^2}{\omega^3}, \quad (54)$$

where

$$v_{eff}^{(\Omega)} = \frac{4}{3} \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{1/2}} \ln \frac{(\kappa T)^{1/2}}{\gamma \Omega_e \sqrt{2m} |ee_i|}, \quad (55)$$

$$\delta v_{\parallel}(\omega) = 2 \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{1/2}} \ln \left| \frac{\Omega_e}{2\omega} \right|, \quad (56)$$

$$\delta v_{\perp}(\omega) = \frac{\sqrt{2\pi} (ee_i)^2 N_i}{\sqrt{m} (\kappa T)^{1/2}} \begin{cases} \left[\ln \frac{\Omega_e}{\omega} \right]^2, & \Omega_e \sqrt{m/m_i} < \omega < \Omega_e; \\ \left[\ln \frac{4m_i}{m} - 1 \right] \ln \left| \frac{\Omega_e}{\omega} \right| - \frac{1}{4} \left[\ln \frac{m_i}{m} \right]^2, & \Omega_i < \omega < \Omega_e \sqrt{m/m_i}; \\ \left[\ln \frac{\Omega_i}{\omega} \right]^2 + \frac{3}{4} \left[\ln \frac{m_i}{m} \right]^2 + \ln \frac{m_i}{m} \ln \left| \frac{\Omega_i}{\omega} \right|, & \omega < \Omega_i \end{cases} \quad (57)$$

$$(58)$$

$$(59)$$

A comparison of Eqs. (54) – (59) with Eq. (34) indicates that in strong fields there are two effective collision frequencies or relaxation times.* If $\omega \sim \omega_{Le}$ in Eq. (57) this equation contains a higher order term which, together with Eq. (55), leads to an expression for the transverse relaxation time; the transverse relaxation time gives the coefficient for electron-ion diffusion across the magnetic field.^[5]

*Equation (20) of^[4] can be regarded as an interpolation formula which gives Eqs. (18) and (15) of the present work as limiting cases. When $\omega \sim \Omega_e$ the approximate kernel of the collision integral, which corresponds to Eq. (17) of^[4] is a poor approximation.

In all formulas in this paper we have assumed that the maximum frequency ($\omega_{\max}$) is determined by the limit of applicability of perturbation theory. However, if this restriction does not hold and instead the range of applicability of our formulas in the region of small impact parameters is determined by quantum effects in the kinetic equation (1), which have been neglected, then $\omega_{\max}$ is replaced by $(\kappa T/\hbar)$.

I am indebted to V. L. Ginzberg for his interest in this work.

¹V. P. Silin, JETP **38**, 1771 (1960), Soviet Phys. JETP **11**, 1277 (1960).

²V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in Plasma) Fizmatgiz, 1960.

³A. V. Gurevich, JETP **35**, 392 (1958), Soviet Phys. JETP **8**, 271 (1959).

⁴A. A. Rukhadze and V. P. Silin, Elektromagnitnye svoïstva plazmy i plazmopodobnykh sred (Electromagnetic Properties of Plasma and Plasma Media) Atomizdat, 1961.

⁵S. T. Belyaev, Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsii (Plasma Physics and the Problem of a Controlled Thermo-nuclear Reaction), AN SSSR, 1958, Vol. 3, p. 66.

Translated by H. Lashinsky
149

FISSIONABILITY OF NUCLEI BY HIGH ENERGY PROTONS

N. A. PERFILOV

Radium Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 5, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 871-873 (September, 1961)

Experimental data on fission of nuclei induced by high energy protons are analyzed and the relation between the limiting value of the fission cross section σ_f and the parameter Z^2/A is established.

FROM consideration of the dependence of the cross section σ_f for the fission of U^{238} , Th^{232} , and Bi^{209} on the proton energy (see [1]) it follows that at a sufficiently high proton energy the value of the ratio σ_f/σ_t attains a value which, within the limits of experimental error, does not depend on the further increase in the proton energy.

The total cross section of the inelastic interaction σ_t for a proton energy $E_p \sim 300$ Mev and higher can be estimated [2] from the expression

$$\sigma_t = \pi (aA^{1/3} + r')^2 \cdot 10^{-26} \text{ cm}^2,$$

where A is the mass number, the constant a equals 1.26 cm and $r' = -0.41$ cm.

For uranium and thorium σ_f/σ_t becomes independent of the energy when $E_p > 100$ Mev, and the values of σ_f at $E_p = 300$ Mev are equal to 1.3 and 0.8 b, respectively. [1] For 9-Bev protons the fission cross section for uranium is also ~ 1.3 b. [3]

For bismuth and lead [1,4,5] the maximum measured fission cross sections are ~ 0.2 and 0.1 b. The fissionability attains saturation at a proton energy ~ 350 Mev.

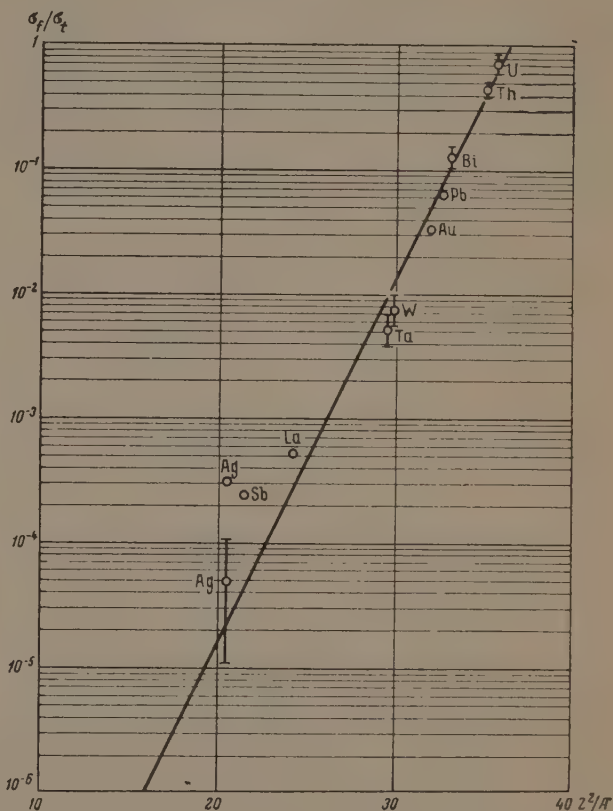
The fission cross section for Au^{197} has been measured up to $E_p \approx 350$ Mev, [1] and at this energy is 0.05 b. The fissionability at 350 Mev does not yet attain the maximum value, but is apparently close to it.

For tungsten and tantalum σ_f , at $E_p = 660$ Mev, is 11 ± 3 and 8 ± 3 mb, respectively, according to measurements by photographic [6] and radio-metric [7] methods.

La^{134} and Sb^{122} at $E_p = 660$ Mev have fission cross sections of 0.6 and 0.25 mb. [8,9]

The fission cross section for silver is 0.3 mb according to Shamov, [10] and about 0.05 mb according to Kofstad. [11]

In the figure, values of the fissionability σ_f/σ_t are shown on a semilogarithmic scale as a function of Z^2/A . It is seen that the experimental



points do not lie smoothly on a straight line. Three values of the fissionability (for lanthanum, antimony, and silver, data from [10]) are far from the line. Perhaps in this region of nuclei a fission mechanism occurs which is different from the classical one, or there is a large contribution from fragmentation processes in the experimentally observed cross sections. The latter is probable, since the cross section seems to increase with a decrease in Z .

If it assumed that the line correctly reflects the dependence of σ_f/σ_t on Z^2/A , then this dependence can be written in analytic form:

$$\sigma_f/\sigma_t = \exp \{0.682 [Z^2/A - 36.25]\},$$

from which it follows that for $Z^2/A \approx 36.25$ the

inelastic cross section should be entirely determined by fission.

The possibility of representing the dependence of the ratio σ_f/σ_t on Z^2/A in exponential form over a large interval of values of Z is, perhaps, an indication that there is only one fission mechanism in this interval.

¹ H. M. Steiner and J. A. Jungerman, *Phys. Rev.* **101**, 807 (1956).

² Millburn, Birnbaum, Crandall, and Schecter, *Phys. Rev.* **95**, 1268 (1954).

³ Perfilov, Darovskikh, Denisenko, and Obukhov, *JETP* **38**, 716 (1960), *Soviet Phys. JETP* **11**, 517 (1960).

⁴ Vinogradov, Alimarin, Baranov, Lavrukhin, Baranova, and Pavlovskaya, *Sessiya AN SSSR po mirnomu ispol'zovaniyu atomnoĭ énergii*, Otd. Khim. Nauk, (Session of the Acad. Sci., U.S.S.R.

on the Peaceful Uses of Atomic Energy, Chem. Sci. Sec.) July, 1955, AN SSSR, 1955.

⁵ Wolfgang, Baker, Caretto, Cumming, Friedlander, and Hudis, *Phys. Rev.* **103**, 394 (1956).

⁶ Perfilov, Ivanova, Lozhkin, Ostroumov, and Shamov, loc. cit. [⁴].

⁷ Baranovskii, Murin, and Preobrazhenskii, *Radiokhimiya* (Radiochemistry) (in press).

⁸ Lavrukhina, Krasavina, and Pozdnyakov, *DAN SSSR* **119**, 56 (1958), *Soviet Phys.-Doklady* **3**, 283 (1958).

⁹ Lavrukhina, Rakowski, Su, and Chojnacki, *JETP* **40**, 409 (1961), *Soviet Phys. JETP* **13**, 280 (1961).

¹⁰ V. P. Shamov, *JETP* **35**, 316 (1958), *Soviet Phys. JETP* **8**, 219 (1959).

¹¹ K. Kofstad, UCRL-2265, 1953.

Translated by E. Marquit
150

SINGLE PARTICLE EXCITED STATES AND THE MODEL DESCRIPTION OF LIGHT NUCLEI

V. G. NEUDACHIN and V. N. ORLIN

Nuclear Physics Institute, Moscow State University

Submitted to JETP editor April 7, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 874-876 (September, 1961)

We show that the location and reduced widths of the positive parity levels in $C^{13} - N^{13}$ are in good agreement with the unified model. The reason for the small separation between the ground state of $C^{13} - N^{13}$ and the closest single-particle levels with positive parity is the large nuclear deformation. The general features of the giant dipole resonance are in good agreement with the picture of a strong deformation in the light nuclei. The suggestion is made that in O^{17} the outer nucleon moves in the field of a highly nonspherical core with tetrahedral symmetry. Possibly the most generally suitable description of the collective properties of light nuclei is given by the α model, where in many cases ($C^{13} - N^{13}$, $Mg^{25} - Al^{25}$, etc.) the description of the single-particle levels in the framework of the α model may be close to the description on the unified model.

FROM an analysis of the single-particle excited states of p-shell nuclei, in particular $C^{13} - N^{13}$, Lane^[1,2] proposed the hypothesis of a weak coupling of the 2s and 1d nucleons with the rest of the nucleus (the 1p core). We want to point out various objections to this point of view.

In $C^{13} - N^{13}$ the separation between the single-particle level of the p shell (ground state) and the nearest single-particle levels of the 1d-2s shell is 3-4 Mev, which is approximately one fourth of the value obtained from electron scattering by C^{12} (where $\hbar\omega = 16$ Mev was used^[3] in describing C^{12} by means of harmonic oscillator functions on the shell model). A similar situation exists for the other well-investigated $Mg^{25} - Al^{25}$ nucleus. It is known^[4] to have a sizable deformation ($\delta = +0.3-0.4$, where the notation is that of Nilsson^[5]) and therefore the distance between the ground state, in which the odd nucleon is in the 1d-2s shell, and the nearest levels of the 1f-2p shell is ~ 4 Mev, even though $\hbar\omega_0^0 = 13$ Mev, as is indicated by the electron scattering data.^[3] It is natural to attempt an explanation of the features of the levels of C^{13} on a similar basis.

A comparison of the wave functions for C^{12} on the shell model and on the unified model shows that the description of C^{12} on the unified model is very close to the shell model description if the nucleus is highly oblate.^[6] Taking $\hbar\omega_0^0 = 16$ Mev (from electron scattering^[3]) and $C = -2$ Mev (from the spin-orbit splitting in $O^{17} - F^{17}$ and $Ca^{41} - Sc^{41}$)^[7,8] and using Nilsson's results,^[5] we get the correct positions of all the known

positive-parity levels of $C^{13} - N^{13}$ as well as their reduced nucleon widths for a value of $\delta \approx -0.6$. In the computations the moment of inertia was found from the position of the 2^+ level of C^{12} . Corrections for coupling of the motion of the external nucleon to the rotation amount to 0.2-0.3 Mev, so that they were omitted. The results for $\delta = -0.65$ are shown in the table. The ground state of $C^{13} - N^{13}$ is orbit 4.

In addition to the reduced widths θ^2 given in the table, θ^2 was computed for the s-nucleon decay of the 6.91 (7.64) Mev level of $C^{13} (N^{13})$ to the 2^+ level of C^{12} . The computation gave $\theta^2 = 0.35$, in approximate agreement with experiment.^[9] Such a picture also gives completely satisfactory agreement with the general features of the giant resonance dipole absorption of γ quanta in Li^7 , $C^{12} - C^{13}$ and $Mg^{24} - Mg^{25}$, including the absolute position of the maximum, which is calculated by the method of ref. 10: for Li^7 , in accordance with the work of Inopin et al.^[11] and Kurath and Pičman,^[6] large values of the deformation are used.

Although the computations were done with wave functions containing "superfluous" coordinates, a comparison of a whole series of specific results with the results of computations made with exact functions shows that they are in good agreement, if, as one should have anticipated, the Bohr-Mottelson functions are taken with a somewhat smaller deformation parameter than the exact functions. However, it remains a mystery why in $O^{17} - F^{17}$ the lowest single-particle levels of the

Positive parity levels of C^{13}

Orbits	5	6	6	6	7	8	9	11
$\Omega\pi$	$5/2^+$	$1/2^+$	$1/2^+$	$1/2^+$	$3/2^+$	$3/2^+$	$1/2^+$	$1/2^+$
J	$5/2$	$1/2$	$3/2$	$5/2$	$3/2$	$3/2$	$1/2$	$1/2$
$E_{exc}(\text{Mev})$ {theor.	3.5	3.0	8.7	6.7	7.5	13	15	23
{exp.	3.85	3.09	7.64	6.86	8.3			
θ^2 {theor.	0.35	0.35	0.04	0.08	0.30			
{exp.	0.2	0.5	0.04	0.01	0.5			

1f—2p shell also appear, starting with an excitation of 4—5 Mev. For example, the $3/2^-$ level at 4.55 (4.4) Mev in O^{17} (F^{17}) has a reduced width which is approximately the same as that of the $1d_{5/2}$ and $1d_{3/2}$ levels.^[1,7,12,13]

It seems to us that the only description of all the above facts from a single viewpoint is the description by means of the α model. The α model immediately shows that in Be^9 the even-even core in whose field the odd nucleon moves, is highly prolate, whereas it is highly oblate in C^{13} . Moreover the α model predicts that the deviation from spherical shape is much greater for Ne^{21} than for Mg^{25} . In fact, for Ne^{21} , $\delta = 0.5-0.6$, if the 4.81 Mev level is the ground level of orbit 7 (on the basis of the experimental data for Ca^{41} ,^[8] we set $D = 0$ for the 1f—2p shell). The success of the unified model in describing Be^9 , C^{13} , Mg^{25} and other nuclei may be related to the fact that for these nuclei it is a good approximation to replace the α core by an ellipsoid. Only for the nuclei near O^{16} will this approximation be unsuccessful (tetrahedral nuclei), and here even for a crude description one needs a different mathematical apparatus from that of the unified model (cubic harmonics). The small separation of the different shells in O^{17} shows that there is a marked deviation of the core from spherical shape, i.e., there is a strong coupling of the nucleon with the deformed (tetrahedral) core. Another indication of the strong coupling is the fact that the reduced widths of all the single-particle levels of $O^{17} - F^{17}$, excepting possibly the $2s_{1/2}$ level, are two to three times smaller than the Wigner limit. (This same situation is also characteristic for the unified model when there is strong coupling of the nucleon to the core.^[14])

In conclusion we emphasize that the experimental data show that the nonadiabatic coupling of the external nucleon to the core vibrations plays an important part in electromagnetic transitions (whereas it has little effect on level positions). Thus in O^{17} the quadrupole moment corresponds to an "effective charge" of the neutron of ~ 0.5 ,^[15] one can draw a similar conclusion for

C^{13} on the basis of recent results of Wilkinson et al.^[16] for the intensity of the 3.85 Mev $\rightarrow$ 3.09 Mev E2 transition in C^{13} . The appearance of the α model vibrations ($T = 0$) in this case is indicated by the fact that E1 transitions between the low-lying states of $C^{13} - N^{13}$ seem to be quite strongly inhibited (by a factor $\sim 10^{-2}$ compared to a Weisskopf unit^[2]). This is explained qualitatively by the selection rules for the asymptotic quantum numbers in the unified model,^[2] although quantitatively the unified model gives a very much stronger inhibition ($10^{-4} - 10^{-5}$).^[2] Such a reduction of the inhibition is not surprising in view of the large effect of the nonadiabaticity on the radiation widths and the differences between the wave functions of the α model and the unified model.

¹ A. M. Lane, Revs. Modern Phys. **32**, 519 (1960).

² D. Kurath and R. Lawson, Nuclear Phys. **23**, 5 (1961).

³ R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

⁴ A. E. Litherland et al., Can. J. Phys. **36**, 378 (1958); Litherland, Gove, and Ferguson, Phys. Rev. **114**, 1312 (1959).

⁵ S. G. Nilsson, Kgl. Danske Videnskab. Selskab. Mat.-Fys. Medd. **29**, No. 16 (1955).

⁶ D. Kurath and L. Pičman, Nuclear Phys. **10**, 313 (1959).

⁷ F. Ajzenberg and T. Lauritsen, Nuclear Phys. **11**, 1 (1959).

⁸ Class, Davis, and Johnson, Phys. Rev. Letters **3**, 41 (1959).

⁹ Reich, Phillips, and Russell, Phys. Rev. **104**, 143 (1956).

¹⁰ Neudachin, Shevchenko, and Yudin, JETP **39**, 108 (1960), Soviet Phys. JETP **12**, 79 (1961); Proc. Int. Conference on Nuclear Structure, Kingston, 1960, p. 732.

¹¹ Gonchar, Inopin, and Tsytko, International Conference on Nuclear Physics, Paris, 1958.

¹² M. H. Macfarlane and J. B. French, Revs. Modern Phys. **32**, 567 (1960).

¹³ E. L. Keller, Phys. Rev. **121**, 820 (1961).

¹⁴ S. Yoshida, Progr. Theor. Phys. **12**, 141 (1954);

J. R. Satchler, Annals of Physics **3**, 275 (1958).

¹⁵ Kamper, Lea, and Lustig, Proc. Phys. Soc. (London) **B70**, 897 (1957).

¹⁶ Pixley, Kane, and Wilkinson, Phys. Rev. **120**, 943 (1960).

Translated by M. Hamermesh
151

THE UPPER CRITICAL FIELD OF SUPERCONDUCTING ALLOYS

E. A. SHAPOVAL

Moscow State University

Submitted to JETP editor April 7, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 877-885 (September, 1961)

The value of the upper critical field H_{C1} of superconducting alloys is derived for the case when the mean free path is small compared with the parameter $\xi = v/\Delta$ of the Bardeen, Cooper, and Schrieffer theory of superconductivity. The temperature variation of H_{C1} is nearly linear, and the second derivative with respect to temperature is positive.

It is well known that superconducting alloys have a number of anomalous properties. In particular, the transition from the normal to the superconducting state in a magnetic field starts at some "upper" critical field H_{C1} exceeding the thermodynamic critical field H_C , —the transition being a second order phase transition. From the viewpoint of the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity^[1] this means that, at fields smaller than the upper critical field, the electron interactions through the phonon field lead to the formation of Cooper pairs and the appearance of a gap in the energy spectrum. As $H \rightarrow H_{C1}$ from below, the gap tends to zero, and for $H > H_{C1}$ there are no solutions with a gap. Thus, the upper critical field of superconducting alloys is analogous to the critical field of supercooled, impurity-free superconductors (which has been considered by Gor'kov^[2]), in the sense that it is also determined as the stability limit of the normal phase with respect to the appearance of superconducting correlations. In the present paper the value of the field H_{C1} for superconducting alloys is obtained for the whole temperature range, when the mean free path is small in comparison with the parameter $\xi_0 = v/\Delta$ of the BCS theory.

To begin with we consider the absolute zero of temperature. We write the equations of the theory of superconductivity in the form used by Gor'kov.^[3] When impurities are present and there is a constant magnetic field, they are

$$\left\{ \omega + \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}} - i \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}) + \mu \right\} G_{\omega}(\mathbf{r}, \mathbf{r}') + i\Delta(\mathbf{r}) F_{\omega}^{+}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (1)$$

$$\left\{ -\omega + \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}} + i \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}) + \mu \right\} F_{\omega}^{+}(\mathbf{r}, \mathbf{r}') + i\Delta^{*}(\mathbf{r}) G_{\omega}(\mathbf{r}, \mathbf{r}') = 0, \quad (2)$$

$$\Delta^{*}(\mathbf{r}) = (2\pi)^{-1} |g| \int_{-\infty}^{+\infty} F_{\omega}^{+}(\mathbf{r}, \mathbf{r}) d\omega, \quad (3)$$

$A_x = -Hy$, $A_y = A_z = 0$ (the magnetic field is taken along the z axis); $V(\mathbf{r})$ is the potential energy of interaction with all the impurity atoms

$$V(\mathbf{r}) = \sum_a u(\mathbf{r} - \mathbf{r}_a), \quad (4)$$

where the sum is taken over all the impurity atoms randomly distributed throughout the lattice.

We need to find the field at which a solution of equations (1), (2), (3) with vanishingly small $\Delta(\mathbf{r})$ and $F_{\omega}^{+}(\mathbf{r}, \mathbf{r}')$ first appears as the magnetic field decreases. The system is thus greatly simplified, since we can limit ourselves to first order terms in $\Delta(\mathbf{r})$ and $F_{\omega}^{+}(\mathbf{r}, \mathbf{r}')$. Then, instead of (2), we have

$$\left\{ -\omega + \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}} + i \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}) + \mu \right\} F_{\omega}^{+}(\mathbf{r}, \mathbf{r}') + i\Delta^{*}(\mathbf{r}) \tilde{G}_{\omega}(\mathbf{r}, \mathbf{r}') = 0, \quad (5)$$

where $\tilde{G}_{\omega}(\mathbf{r}, \mathbf{r}')$ is the Green's function of an electron in the normal metal when impurities are present and there is a magnetic field; this satisfies the equation

$$\left\{ \omega + \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}} - i \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 + V(\mathbf{r}) + \mu \right\} \tilde{G}_{\omega}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \quad (6)$$

or

$$\left\{ \omega + \frac{1}{2m} \left(\frac{\partial}{\partial \mathbf{r}'} + i \frac{e}{c} \mathbf{A}(\mathbf{r}') \right)^2 + V(\mathbf{r}') + \mu \right\} \tilde{G}_{\omega}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (6')$$

Then,

$$F_{\omega}^{+}(\mathbf{r}, \mathbf{r}') = -i \int \tilde{G}_{\omega}(\mathbf{r}'', \mathbf{r}') \tilde{G}_{-\omega}(\mathbf{r}'', \mathbf{r}) \Delta^{*}(\mathbf{r}'') d\mathbf{r}''. \quad (7)$$

Using (3), we arrive at the integral equation

$$\Delta^{*}(\mathbf{r}) = -i(2\pi)^{-1} |g| \int_{-\infty}^{+\infty} d\omega \int \tilde{G}_{\omega}(\mathbf{r}', \mathbf{r}) \tilde{G}_{-\omega}(\mathbf{r}', \mathbf{r}) \Delta^{*}(\mathbf{r}') d\mathbf{r}'. \quad (8)$$

(2) The asterisk signifies averaging over all possible

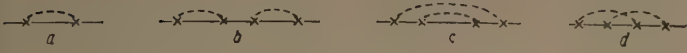


FIG. 1

positions of the impurity atoms. The maximum field for which there is a non-zero solution of this integral equation is the upper critical field sought.

The averaging of the Green's functions over the impurity atom positions is performed using the diagram technique developed by Abrikosov and Gor'kov^[4,5] and by Edwards.^[6] Each scattering of an electron at an impurity atom contributes a factor $u(\mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{r}_a)$ to the expression for the Green's function in the momentum representation, where $\mathbf{q}$ is the momentum transmitted, and $u(\mathbf{q})$ is the Fourier component of the impurity atom potential. On averaging over $\mathbf{r}_a$, this factor becomes zero unless there are terms in the averaged expression corresponding to scattering at the same impurity atom, but with $\mathbf{q}' = -\mathbf{q}$; in this case a multiplier $|u(\mathbf{q})|^2$ occurs on averaging, which is proportional to the scattering probability in the Born approximation with transmitted momentum $\mathbf{q}$. It is thus necessary to pick out in pairs all scattering events at the same atoms.

In the diagram each scattering is indicated by a cross; averaging is indicated by joining two crosses corresponding to scattering at the same atom by a broken line. In Fig. 1 several such diagrams are shown. On summing in diagrams with intersecting broken lines (for example, diagram d), the integration is over a region far from the Fermi surface, and they therefore make an insignificant contribution ($\sim 1/p_0 l$, where p_0 is the Fermi momentum, and $l = v\tau$ is the mean free path), which can be neglected. Summation of the remaining diagrams leads, as usual, to an integral equation.

Applying the technique described to the Green's function of a normal metal in the absence of a magnetic field, we have

$$G_\omega(\mathbf{p}) = (\omega - \xi_{\mathbf{p}} + i \operatorname{sign} \omega / 2\tau)^{-1}, \quad (9)$$

$$G_\omega(\mathbf{R}) = -\frac{m}{2\pi R} \exp \left\{ iR \left(p_0 + \frac{\omega}{v} \right) \operatorname{sign} \omega - \frac{R}{2v\tau} \right\}, \quad (10)$$

$$\frac{1}{\tau} = \frac{nm p_0}{(2\pi)^3} \int |u(\mathbf{q})|^2 d\mathbf{q} \quad (11)$$

(n is the concentration of impurity atoms). In a homogeneous magnetic field along the z axis (if



FIG. 3

the curvature of the electron trajectories is small, which is the case in the fields of interest to us), we have

$$\tilde{G}_\omega(\mathbf{r}, \mathbf{r}') = \exp \{ -i(eH/c)(y + y')(x - x') \} G_\omega(\mathbf{r} - \mathbf{r}'). \quad (12)$$

The product of two Green's functions is averaged similarly. The lowest order diagrams arising from averaging in pairs are shown in Fig. 2. The diagrams with intersecting broken lines (for example, d and e) are unimportant; the broken lines joining crosses sited on one electron line (for example, diagram f) give complete Green's functions for this line. Thus, we must sum the "ladder" diagrams shown in Fig. 3. We denote the sum of these diagrams by

$$\Gamma_\omega(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2') = \tilde{G}_\omega(\mathbf{r}_1, \mathbf{r}_1') \tilde{G}_{-\omega}(\mathbf{r}_2, \mathbf{r}_2'). \quad (13)$$

It is not difficult to write down the integral equation for $\Gamma_\omega(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$. From this equation (which we do not write down because it is very cumbersome) it is possible to extract an exponential factor caused by the superposition of the magnetic field, and to transform the equation from coordinate to momentum space, i.e., to express $\Gamma_\omega(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$ as:

$$\Gamma_\omega(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2') = \exp \{ -i(eH/2c)(y_1 + y_1')(x_1 - x_1') + (y_2 + y_2')(x_2 - x_2') \} \Gamma_\omega(\mathbf{r}_1 - \mathbf{r}_1'; \mathbf{r}_2 - \mathbf{r}_2'). \quad (14)$$

We introduce the Fourier components

$$\Gamma_\omega(\mathbf{r}, \mathbf{r}') = (2\pi)^{-6} \iint e^{i\mathbf{p}\mathbf{r} + i\mathbf{p}'\mathbf{r}'} \Gamma_\omega(\mathbf{p}, \mathbf{p}') d\mathbf{p} d\mathbf{p}'. \quad (15)$$

They satisfy the equation

$$\begin{aligned} \Gamma_\omega(\mathbf{p}_1, \mathbf{p}_2) &= G_\omega(\mathbf{p}_1) G_{-\omega}(\mathbf{p}_2) \\ &+ \frac{n}{(2\pi)^3} \int |u(\mathbf{q})|^2 G_\omega(\mathbf{p}_1 - \mathbf{m}_1 + 2\mathbf{l}_2) \\ &\times G_{-\omega}(\mathbf{p}_2 - \mathbf{m}_2 + 2\mathbf{l}_2) \Gamma_\omega(\mathbf{p}_1 + \mathbf{q} - \mathbf{m}_1 \\ &+ \mathbf{l}_1, \mathbf{p}_2 - \mathbf{q} - \mathbf{m}_2 + \mathbf{l}_2) \\ &\times \delta_H(\mathbf{m}_1, \mathbf{l}_1) \delta_H(\mathbf{m}_2, \mathbf{l}_2) d\mathbf{m}_1 d\mathbf{m}_2 d\mathbf{l}_1 d\mathbf{l}_2 d\mathbf{q}, \end{aligned} \quad (16)$$

where

$$\delta_H(\mathbf{m}, \mathbf{l}) = (2c/eH)^4 \exp \{ i(2c/eH)(m_x l_y - m_y l_x) \} \delta(m_z) \delta(l_z), \quad (16')$$

$$\delta_H(\mathbf{m}, \mathbf{l}) \rightarrow \delta(\mathbf{m}) \delta(\mathbf{l}) \text{ as } H \rightarrow 0.$$

If we write

$$\begin{aligned} \Gamma'(\mathbf{p}, \mathbf{k}, \omega) &= \frac{n}{(2\pi)^3} \int |u(\mathbf{q})|^2 \Gamma_\omega \left(\mathbf{p} + \frac{\mathbf{k}}{2} + \mathbf{q}, -\mathbf{p} \right. \\ &\left. + \frac{\mathbf{k}}{2} - \mathbf{q} \right) d\mathbf{q}, \end{aligned} \quad (17)$$

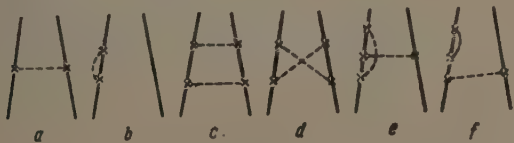


FIG. 2

then

$$\begin{aligned} \Gamma_{\omega}(\mathbf{p}_1, \mathbf{p}_2) &= G_{\omega}(\mathbf{p}_1) G_{-\omega}(\mathbf{p}_2) + \int G_{\omega}(\mathbf{p}_1 - \mathbf{m}_1 \\ &+ 2\mathbf{l}_1) G_{-\omega}(\mathbf{p}_2 - \mathbf{m}_2 + 2\mathbf{l}_2) \Gamma'(\tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{m}_2 - \mathbf{m}_1 \\ &+ \mathbf{l}_1 - \mathbf{l}_2), \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{m}_1 - \mathbf{m}_2 + \mathbf{l}_1 + \mathbf{l}_2, \omega) \\ &\times \delta_H(\mathbf{m}_1, \mathbf{l}_1) \delta_H(\mathbf{m}_2, \mathbf{l}_2) d\mathbf{m}_1 d\mathbf{m}_2 d\mathbf{l}_1 d\mathbf{l}_2, \end{aligned} \quad (18)$$

and $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ satisfies the equation

$$\begin{aligned} \Gamma'(\mathbf{p}, \mathbf{k}, \omega) &= \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 G_{\omega}(\mathbf{p}_1 + \frac{\mathbf{k}}{2}) G_{-\omega}(\mathbf{p}_1 \\ &- \frac{\mathbf{k}}{2}) d\mathbf{p}_1 + \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 G_{\omega}(\mathbf{p}_1 + \frac{\mathbf{k}}{2} \\ &- \mathbf{m}_1 + 2\mathbf{l}_1) G_{-\omega}(\mathbf{p}_1 - \frac{\mathbf{k}}{2} + \mathbf{m}_2 - 2\mathbf{l}_2) \\ &\times \Gamma'(\mathbf{p}_1 - \frac{1}{2}(\mathbf{m}_1 - \mathbf{m}_2 - \mathbf{l}_1 + \mathbf{l}_2), \mathbf{k} - \mathbf{m}_1 - \mathbf{m}_2 \\ &+ \mathbf{l}_1 + \mathbf{l}_2, \omega) \delta_H(\mathbf{m}_1, \mathbf{l}_1) \delta_H(\mathbf{m}_2, \mathbf{l}_2) d\mathbf{p}_1 d\mathbf{m}_1 d\mathbf{m}_2 d\mathbf{l}_1 d\mathbf{l}_2. \end{aligned} \quad (19)$$

In this equation the product of two Green's functions decreases rapidly on departing from the Fermi surface. On the other hand, $u(\mathbf{p} - \mathbf{p}_1)$ depends weakly on $\mathbf{p} - \mathbf{p}_1$, and changes significantly only when the argument is changed by a quantity of the order of the Fermi momentum. Therefore $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ also depends weakly on $|\mathbf{p}|$. By virtue of this we can integrate with respect to $d|\mathbf{p}|$:

$$\begin{aligned} \Gamma'(\mathbf{p}, \mathbf{k}, \omega) &= \frac{inmp_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 \left(2\omega - \frac{\mathbf{p}\mathbf{k}}{m} + \frac{i}{\tau}\right)^{-1} d\omega_1 \\ &+ \frac{inmp_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 \left(2\omega - \frac{\mathbf{p}}{m}(\mathbf{k} - \boldsymbol{\eta}_1 + 2\boldsymbol{\eta}_2) + \frac{i}{\tau}\right)^{-1} \\ &\times \Gamma'(\mathbf{p}_1 - \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, \mathbf{k} - \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2, \omega) \delta_H(\boldsymbol{\eta}_2, \boldsymbol{\xi}_1) \delta_H(\boldsymbol{\xi}_2, \boldsymbol{\eta}_1) \\ &\times d\boldsymbol{\eta}_1 d\boldsymbol{\eta}_2 d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\omega_1 \end{aligned} \quad (20)$$

(the variables have been changed: $\mathbf{m}_1 + \mathbf{m}_2 = \boldsymbol{\eta}_1$, $\mathbf{m}_1 - \mathbf{m}_2 = 2\boldsymbol{\xi}_1$ etc). Integrating now with respect to $d\boldsymbol{\xi}_1$ and $d\boldsymbol{\xi}_2$, we obtain a δ -type function of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, the width being, in order of magnitude, $eH/cp_0 = R_H^{-1}$ (R_H is the radius of curvature of the electron trajectories in the magnetic field). If $2\omega - \mathbf{p}\mathbf{k}/m + i/\tau$ and $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ change little on changing $\mathbf{k}$ by the order of R_H^{-1} , we can integrate with respect to $d\boldsymbol{\eta}_1$ and $d\boldsymbol{\eta}_2$:

$$\Gamma'(\mathbf{p}, \mathbf{k}, \omega) = \frac{inmp_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 \frac{1 + \Gamma'(\mathbf{p}_1, \mathbf{k}, \omega)}{2\omega - \mathbf{p}_1\mathbf{k}/m + i/\tau} d\omega_1. \quad (21)$$

The applicability of this equation is limited by two conditions:

$$\partial\Gamma'(\mathbf{p}, \mathbf{k}, \omega)/\partial\mathbf{k} \ll \Gamma'(\mathbf{p}, \mathbf{k}, \omega) R_H, \quad (22)$$

$$v\tau \ll R_H, \quad (23)$$

the second is satisfied for the fields considered; a study of Eq. (21) shows that the first condition is violated when $\mathbf{k} \lesssim R_H^{-1}$. Such values of $\mathbf{k}$ correspond to distances $\gtrsim R_H$ at which the quasi-classical approximation for the Green's function

in a magnetic field becomes invalid. In the problem under consideration, as we will see below, smaller distances are important, in fact, distances $\sim \sqrt{c/eH} = \sqrt{R_H/p_0}$.

By a similar line of reasoning, it can be shown that, when conditions (22) and (23) are satisfied, we have, instead of (18):

$$\begin{aligned} \Gamma_{\omega}(\mathbf{p}_1, \mathbf{p}_2) &= G_{\omega}(\mathbf{p}_1) G_{-\omega}(\mathbf{p}_2) \left[1 + \Gamma'(\tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2), \mathbf{p}_1 + \mathbf{p}_2, \omega) \right]. \end{aligned} \quad (24)$$

We return to the initial equation of the problem [Eq. (18)], rewritten as:

$$\Delta^*(\mathbf{r}) = \int \exp\left\{-i\frac{eH}{2c}(\mathbf{y} + \mathbf{y}')(\mathbf{x} - \mathbf{x}')\right\} K(\mathbf{r} - \mathbf{r}') \Delta^*(\mathbf{r}') d\mathbf{r}'. \quad (25)$$

From (13) – (15) and (24) we obtain, using the above-mentioned properties of $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ and of the product of two G functions,

$$\begin{aligned} K(\mathbf{r} - \mathbf{r}') &= -i|g|\pi^{-1} \int_0^{\infty} \Gamma_{\omega}(\mathbf{r} - \mathbf{r}', \mathbf{r} - \mathbf{r}') d\omega \\ &= \frac{-i|g|}{(2\pi)^3 \pi} \int_0^{\infty} d\omega \int e^{ik(\mathbf{r} - \mathbf{r}')\cdot\mathbf{p}} G_{\omega}(\mathbf{p} + \frac{\mathbf{k}}{2}) G_{-\omega}(-\mathbf{p} + \frac{\mathbf{k}}{2}) \\ &\times [1 + \Gamma'(\mathbf{p}, \mathbf{k}, \omega)] d\mathbf{p} d\mathbf{r} \end{aligned} \quad (26)$$

or

$$K(R) = \frac{-i|g|}{(2\pi)^4} \frac{mp_0}{R} \int_0^{\infty} d\omega \int e^{ikRk} dk \int \frac{1 + \Gamma'(\mathbf{p}, \mathbf{k}, \omega)}{2\omega - \mathbf{p}\mathbf{k}/m + i/\tau} d\omega, \quad (26')$$

$$K(R) = -\frac{4|g|}{(2\pi)^4} \frac{mp_0\tau}{R} \int_0^{\infty} d\omega \int_{-\infty}^{+\infty} \Gamma_0(k, \omega) e^{ikRk} dk; \quad (26'')$$

$$\Gamma_0(k, \omega) = \frac{1}{4\pi} \int \Gamma'(\mathbf{p}, \mathbf{k}, \omega) d\omega = \frac{i}{4\pi} \int \frac{1 + \Gamma'(\mathbf{p}, \mathbf{k}, \omega)}{2\omega - \mathbf{p}\mathbf{k}/m + i/\tau} d\omega. \quad (27)$$

The last equation can be obtained by integrating both parts of (21) with respect to $d\omega$ (i.e., with respect to the directions $\mathbf{p}$), using (11).

Equation (21) can be solved in the limiting case $\mathbf{k} \ll l^{-1}$. Then

$$\Gamma_0(k, \omega) = i(2\omega\tau + \frac{1}{3}ik^2v^2\tau\tau_{tr})^{-1}, \quad (28)$$

$$\frac{1}{\tau_{tr}} = \frac{nm p_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 (1 - \cos\theta) d\omega. \quad (29)$$

Whence, for $R \gg l$, we have

$$K(R) = \frac{mp_0|g|}{(2\pi)^3} \frac{2}{R^3}. \quad (30)$$

It is of interest to compare this quantity with the behavior of $K(R)$ at distances small in comparison with the mean free path. In this limiting case values of $\mathbf{k} \gg l^{-1}$ are important. For such values of $\mathbf{k}$ the behavior of $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ is like $(kl)^{-1} \ln(kl)$, and it can be neglected in comparison with unity. Then

$$\Gamma_{\omega}(\mathbf{p}_1, \mathbf{p}_2) = G_{\omega}(\mathbf{p}_1) G_{-\omega}(\mathbf{p}_2),$$

and for $K(R)$ we obtain the same equation as for an impurity-free metal:

$$K(R) = \frac{mp_0 |g|}{(2\pi)^3} \frac{1}{R^3}. \quad (31)$$

We consider the case of a heavily alloyed superconductor, when the mean free path is much smaller than the parameter ξ_0 of the BCS theory. As we shall see below, there exists in this limiting case a solution of (8) or (25), which changes significantly in distances of the order $\sqrt{l\xi_0} \gg l$. The eigenfunctions of this equation, which change rapidly, correspond to small values of the magnetic field. Since we are interested in the maximum field for which a solution exists, $\Delta^*(\mathbf{r})$ can be considered constant at distances of the order of the mean free path. Then, taking into account the behavior of $K(R)$ at large distances, we rewrite (25) as

$$\Delta^*(\mathbf{r}) \left\{ 1 - \int \left(K(\mathbf{r} - \mathbf{r}') - \frac{2mp_0 |g|}{(2\pi)^3 |\mathbf{r} - \mathbf{r}'|^3} \right) d\mathbf{r}' \right\} = \frac{mp_0 |g|}{4\pi^3} \int \frac{\exp \{ ieH(y+y')(x-x')/c \}}{|\mathbf{r} - \mathbf{r}'|^3} \Delta^*(\mathbf{r}') d\mathbf{r}'. \quad (32)$$

The right- and left-hand sides of this equation diverge differently for small values of R . This is due to the fact that we have not taken into account the cutoff of the interaction between the electrons in Bardeen's model Hamiltonian at Debye energies. In our problem this implies that the integration with respect to dp in (26) is performed not between infinite limits, but from $p_0 - 2\tilde{\omega}/v$ to $p_0 + 2\tilde{\omega}/v$. In (26') there appears then in the expression under the integral a multiplier

$$\frac{1}{\pi i} \ln \frac{\tilde{\omega} - \omega + \mathbf{p}\mathbf{k}/2m - i/2\tau}{-\tilde{\omega} - \omega + \mathbf{p}\mathbf{k}/2m - i/2\tau},$$

The expression (31) for $K(R)$ at small values of R is valid only for $R \gg v/\tilde{\omega}$, and the volume integral of $K(R)$ will converge for small values of R .

Performing now the volume integration in the left-hand side of (32) to $|\mathbf{r} - \mathbf{r}'| = \epsilon (\epsilon \rightarrow 0)$, we have

$$\Delta^*(\mathbf{r}) \ln \left(\frac{\tau_{tr} v^2 e^2}{3\gamma^2 \Delta_0 \epsilon^4} \right) = \frac{1}{2\pi} \int_{|\mathbf{r} - \mathbf{r}'| \gg \epsilon} \frac{\exp \{ ieH(y+y')(x-x')/c \}}{|\mathbf{r} - \mathbf{r}'|^3} \Delta^*(\mathbf{r}') d\mathbf{r}'. \quad (33)$$

Here, $\gamma = \exp C$, where C is Euler's constant.

An analogous equation was studied by Gor'kov.^[2] He showed that the solution of this equation can be obtained from the particular solution for $\Delta^*(\mathbf{r})$, depending only on y . In this case

$$\Delta^*(y) \ln \left(\frac{3\gamma^2 \Delta_0}{\tau_{tr} v^2} \epsilon^2 \right) = - \int_{|y-y'| \gg \epsilon} \frac{\exp \{ -eH|y^2 - y'^2|/c \}}{|y - y'|} \Delta^*(y') dy'. \quad (34)$$

Using a variational method [$\exp(-\alpha y^2)$ is taken as the varied function] we obtain the maximum field at zero temperature (which is attained when $\alpha = eH/c$):

$$eH_{c10}/c = 3\gamma \Delta_0 / 2v^2 \tau_{tr}. \quad (35)$$

We introduce the constant κ of the phenomenological Ginzburg-Landau theory,^[9] defined as

$$\kappa = \sqrt{2} 2eH_c \delta^2 / \hbar c$$

as $T \rightarrow T_C$ (H_C is the thermodynamic critical field, δ is the penetration depth). In the limiting case considered,^[5]

$$\kappa = \frac{3}{2\pi^2} \frac{mc}{e\tau_{tr}} \left(\frac{2\pi m}{p_0^5} 7\zeta(3) \right)^{1/2}.$$

The theoretical value of the thermodynamic critical field of a pure metal at $T = 0$ is

$$H_{c0} = \Delta_0 \sqrt{2mp_0/\pi}.$$

The ratio of the two critical fields can be expressed in terms of κ :

$$\frac{H_{c10}}{H_{c0}} = \frac{\pi^2 \gamma}{2\sqrt{7\zeta(3)}} \kappa = 3.03\kappa. \quad (36)$$

The upper critical field can be expressed in terms of the critical temperature T_C , the conductivity of the normal metal at low temperatures σ , and the coefficient γ in the linear thermal capacity relationship:

$$H_{c10} = \frac{3}{2\pi} \frac{e\gamma T_C}{\sigma k}, \quad \frac{H_{c10}}{H_{c0}} = e^C \sqrt{\frac{3}{8\pi^3}} \frac{e\gamma^{1/2}}{\sigma k}. \quad (37)$$

If at $T = 0$ the penetration depth $\delta_0 \gg l$ (when $\delta_0 = (2\pi)^{-1} \times c\sqrt{\hbar/\sigma\Delta_0}$, (cf.^[4]), then

$$H_{c10} = \frac{6\pi^2}{e^C} \frac{e\gamma T_C^2 \delta_0^2}{c\hbar}, \quad \frac{H_{c10}}{H_{c0}} = \sqrt{6\pi^3} \frac{e\gamma^{1/2} T_C \delta_0^2}{c\hbar}. \quad (37')$$

For non-zero temperatures the corresponding thermodynamic technique must be used.^[7,8] Then there will appear in all the formulae sums with respect to $\omega_n = \pi T(2n+1)$ in place of integrals with respect to $d\omega$.

At large distances the kernel $K_T(R)$ will, instead of (30), be

$$K_T(R) = \frac{mp_0 |g|}{(2\pi)^3} \frac{2}{R^3} 2 \int_0^\infty e^{-\eta} \sin \eta \operatorname{th} \left(\frac{v^2 \tau_{tr}}{6TR^2} \eta^2 \right) \eta d\eta. \quad (38)*$$

At finite temperatures Eq. (34) becomes

*th = tanh.

$$\Delta^*(y) \ln \left(\frac{3\gamma^2 \Delta_0}{\tau_{tr} v^2} \varepsilon^2 \right) = - \int_{|y-y'| \geq \varepsilon} K_T(y, y') \Delta^*(y') dy',$$

$$K_T(y, y') = \int_0^\infty \ln \left(\frac{\tau_{tr} v^2}{6T} \eta \right) \times \operatorname{Im} \frac{\exp\{-|y-y'|[(y+y')eH/c]^2 - 2i\eta]^{1/2}\}}{[(y+y')eH/c]^2 - 2i\eta]^{1/2}} d\eta. \quad (39)$$

Again we use a variational method, choosing $\Delta^*(y)$ as $\exp(-\alpha y^2)$. The maximum value of the magnetic field is attained when $\alpha = eH/c$, and is determined by the solution of the equation

$$\ln \left(\frac{3\gamma \Delta_0}{2\tau_{tr} v^2} \frac{c}{eH_{c1}} \right) = F \left(\frac{3\pi T}{2\tau_{tr} v^2} \frac{c}{eH_{c1}} \right),$$

$$F(x) = 2x \int_0^\infty \frac{d\eta}{\eta + 2x} \left(\frac{1}{\eta} - \frac{1}{\sinh \eta} \right) \quad (40)^*$$

($F(x) \rightarrow 2x \ln 2$ as $x \rightarrow 0$ and $F(x) \rightarrow \ln x + \pi^2/8x$ as $x \rightarrow \infty$).

It is convenient to write this equation introducing from (35) – (37) the critical field H_{c10} at $T = 0$:

$$\ln(H_{c10}/H_{c1}) = F(TH_{c10}/T_c H_{c1}). \quad (41)$$

For low temperatures ($T \ll T_c$)

$$\frac{H_{c1}}{H_{c10}} = 1 - \frac{T}{T_c} 2 \ln 2, \quad \frac{H_{c1}}{H_c} = \left(1 - \frac{T}{T_c} 2 \ln 2\right) 3.03\kappa. \quad (42)$$

Close to the critical temperature ($T_c - T \ll T_c$)

$$\frac{H_{c1}}{H_{c10}} = \left(1 - \frac{T}{T_c}\right) \frac{8}{\pi^2}, \quad H_c = \frac{4\gamma}{\sqrt{14\zeta(3)}} \left(1 - \frac{T}{T_c}\right) H_{c0},$$

$$\frac{H_{c1}}{H_c} = \sqrt{2}\kappa, \quad (43)$$

which agrees with the result given by the phenomenological Ginzburg-Landau theory.^[9,10] This is natural, since for $T_c - T \ll T_c$ the integral equation (39) can be transformed into the differential equation of the same problem in this theory.

The variation of the upper critical field on temperature is shown in Fig. 4. It is clear from the graph that the variation is almost linear, the curve being convex from below, i.e., $\partial^2 H_{c1}/\partial T^2 > 0$, in

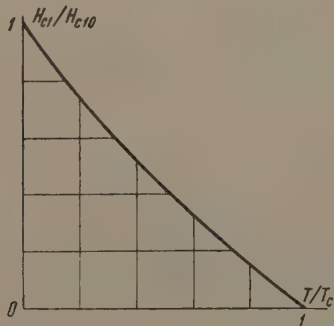


FIG. 4

*sh = sinh.

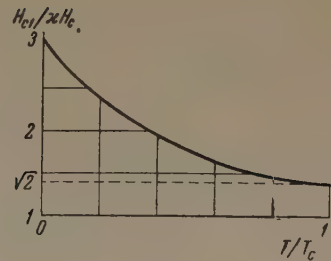


FIG. 5

distinction from the temperature variation of the thermodynamic critical field, which is almost parabolic and has $\partial^2 H_c/\partial T^2 < 0$.

It should be remembered that the variational principle gives a somewhat reduced value of the upper critical field, and the error increases as the temperature is lowered, so that, in fact, the curvature ought to be somewhat greater than that shown in the figure. Unfortunately, there are no experimental data on the temperature variation of the upper critical field throughout a wide temperature range, with which it would have been possible to compare the results of the present work. Experiments close to T_c give a trivial linear variation. For an accurate comparison of the theory with experiment it is necessary to know the mean free path of the electrons, i.e., the residual resistance of the specimens studied.

Figure 5 shows the temperature variation of the ratio H_{c1}/H_c .

In conclusion I am grateful to L. P. Gor'kov for useful discussions and comments, and to L. D. Landau for discussing the results obtained.

¹Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1958).

²L. P. Gor'kov, JETP **37**, 835 (1959), Soviet Phys. JETP **10**, 593 (1960).

³L. P. Gor'kov, JETP **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

⁴A. A. Abrikosov and L. P. Gor'kov, JETP **35**, 1558 (1958), Soviet Phys. JETP **8**, 1090 (1959).

⁵L. P. Gor'kov, JETP **37**, 1407 (1959), Soviet Phys. JETP **10**, 998 (1960).

⁶S. F. Edwards, Phil. Mag. **3**, 1020 (1958).

⁷T. Matsubara, Progr. Theoret. Phys. (Kyoto) **14**, 351 (1955).

⁸Abrikosov, Gor'kov, and Dzyaloshinskii, JETP **36**, 1286 (1959), Soviet Phys. JETP **9**, 912 (1959).

⁹V. L. Ginzburg and L. D. Landau, JETP **20**, 1064 (1950).

¹⁰A. A. Abrikosov, JETP **32**, 1442 (1957), Soviet Phys. JETP **5**, 1174 (1957).

Translated by K. F. Hulme

ABSORPTION OF ELECTROMAGNETIC WAVES IN PLASMA

V. I. PEREL' and G. M. ÉLIASHBERG

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 8, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 886-893 (September, 1961)

The absorption coefficient of electromagnetic waves in a plasma is calculated for $e^2/\hbar v \ll 1$. Application of the temperature diagram technique makes it possible to take systematic account of collective effects and to obtain exact values of the factors in the Coulomb logarithm.

1. As is well known, the Coulomb scattering cross section diverges logarithmically. This leads to the necessity of taking into account the screening, which is generally dynamic.^[1,2] The kinetic plasma coefficients contain the Coulomb logarithm $\ln(r_{\max}/r_{\min})$, where $r_{\max}$ and $r_{\min}$ are respectively the maximum and minimum impact parameters of the collision. The order of magnitude of these quantities can be established from simple physical considerations. For sufficiently low frequencies, the Debye radius can be used for $r_{\max}$.^[3] For frequencies greatly exceeding the plasma frequency, the role of $r_{\max}$ is played by the path traversed by the electrons in a single period.^[4] The value of the parameter $r_{\min}$ depends on the plasma temperature. For high temperatures, when this distance becomes less than the de Broglie wavelength of the electrons, this wavelength can be used for $r_{\min}$.^[5] All these considerations permit us to determine the expression under the logarithm with accuracy up to a factor of the order of unity.

The purpose of the present paper is a rigorous calculation of the absorption coefficient of electromagnetic waves in a high-temperature plasma. An exact value is obtained for the expression under the logarithm for all frequencies of the field that are much higher than the collision frequency (including frequencies of the order of the plasma frequency). A plasma is considered with singly-ionized positive ions.

2. For the determination of the absorption coefficient, one must compute the real part of the electron conductivity of the plasma. We shall start out from the general expression for the conductivity^[2] in the absence of a magnetic field:

$$\sigma_{\mu\nu}(\mathbf{k}, \omega) = \left(\frac{e\hbar}{m}\right)^2 \int p_\mu G_{pp'}(\mathbf{k}, \omega) p'_\nu \frac{d^3 p d^3 p'}{(2\pi)^6}, \quad (1)$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field, ω is its frequency, $\mathbf{p}$ and $\mathbf{p}'$ are the wave

vectors of the electrons, while e and m are the charge and mass of the electron. The function $G_{pp'}(\mathbf{k}, \omega)$ is determined by the relations

$$G_{pp'}(\mathbf{k}, \omega) = \int_0^\infty \exp[(i\omega - \nu)\tau] \tilde{G}_{pp'}(\mathbf{k}, \tau) d\tau, \quad \nu \rightarrow +0, \quad (2)$$

$$\tilde{G}_{pp'}(\mathbf{k}, \tau) = \int_0^{1/T} d\lambda \langle A_{p',k}(\tau - i\hbar\lambda) A_{p,-k}(0) \rangle, \quad (3)$$

where

$$A_{p,k}(z) = \exp(-H'z/i\hbar) a_{p-k/2}^+ a_{p+k/2} \exp(H'z/i\hbar).$$

$H' = H - \mu N$; H is the Hamiltonian of the system, μ is the chemical potential, T is the temperature in energy units, $\langle \dots \rangle$ denotes the statistical average.

For calculation of the function $G_{pp'}(\mathbf{k}, \omega)$, we use the diagram technique suggested by Abrikosov, Gor'kov, and Dzyaloshinskii.^[6] Performing integration by parts in Eq. (2), we get

$$G_{pp'}(\mathbf{k}, \omega) = \{K_{pp'}^R(\mathbf{k}, \omega) - K_{pp'}(\mathbf{k}, 0)\} / i\omega, \quad (4)$$

where

$$K_{pp'}^R(\mathbf{k}, \omega) = \int_{-\infty}^\infty e^{i\omega\tau} \tilde{K}_{pp'}^R(\mathbf{k}, \tau) d\tau, \quad (5)$$

$$K_{pp'}(\mathbf{k}, \omega_n) = \frac{1}{2} \int_{-1/T}^{1/T} e^{i\hbar\omega_n\lambda} \tilde{K}_{pp'}(\mathbf{k}, \lambda) d\lambda, \quad \hbar\omega_n = 2\pi i n T, \quad n = 0, \pm 1, \dots, \quad (6)$$

$$\tilde{K}_{pp'}^R(\mathbf{k}, \tau) = \theta(\tau) \partial \tilde{G}_{pp'}(\mathbf{k}, \tau) / \partial \tau = (i/\hbar) \langle [A_{p',k}(\tau), A_{p,-k}(0)] \rangle \theta(\tau), \quad (7)$$

$$\tilde{K}_{pp'}(\mathbf{k}, \lambda) = \langle T_\lambda(A_{p',k})_\lambda (A_{p,-k})_0 \rangle; \quad (A)_\lambda = e^{\lambda H'} A e^{-\lambda H'}. \quad (8)$$

It can be stated^[6] that the function $K_{pp'}^R(\mathbf{k}, \omega)$ is the analytic continuation of the function $K_{pp'}(\mathbf{k}, \omega_n)$ in the variable ω from discrete points of the imaginary axis ($n > 0$) to the real axis.

3. The function $K_{pp'}(\mathbf{k}, \omega_n)$ is the sum of graphs of which some are shown in Fig. 1. In cor-

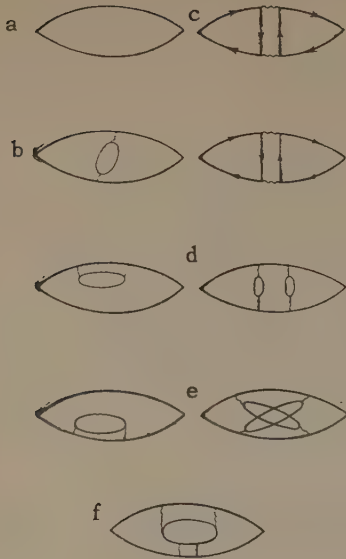


FIG. 1

respondence with the rules of application of the diagram technique,^[6] to each electron line on the graph there corresponds the quantity

$$\frac{1}{2} \int_{-1/T}^{1/T} e^{\lambda e_n} \langle T_\lambda(a_q)_\lambda (a_q^+)_0 \rangle d\lambda = -G_q(e_n) = -(e_n - \epsilon_q)^{-1}, \quad (9)$$

where $e_n = (2n + 1)\pi T$, $\epsilon_q = \hbar^2 q^2 / 2m - \mu$. The interaction matrix element $U_\gamma = 4\pi e^2 / \gamma^2$ corresponds to the wavy line; a minus sign corresponds to each closed loop.

Account of the interaction of the electrons with the ions leads to the appearance of graphs similar to those shown in Fig. 1, on which internal loops are formed by the ion lines. The ion lines will be henceforth drawn dotted. One must integrate over each internal wave vector q in the diagram, so that $(2\pi)^{-3} \int d^3 q$ corresponds to it. Summation is carried out over each internal imaginary "energy" α_n so that $T \Sigma_n$ corresponds to it.

Without account of the interaction between the plasma particles the conductivity is determined by only one graph, shown in Fig. 1a. If the interaction potential were short-range, then we could limit ourselves in the calculation of the absorption coefficient to the graphs shown in Figs. 1b, c and the graphs which are obtained by substitution of the internal electron loop for the ion (see Fig. 1b). It can be expected^[2] that graphs of the type shown in Fig. 1d make a small contribution to the conductivity if $\omega/\nu \gg 1$ (ν is the collision frequency); graphs of the type shown in Fig. 1e are small if the gas condition holds: $e^2 n^{1/3} / T \ll 1$. Finally, one can neglect graphs of the type shown in Fig. 1f if $e^2 / \hbar v \ll 1$ (v is the thermal velocity of the electron).

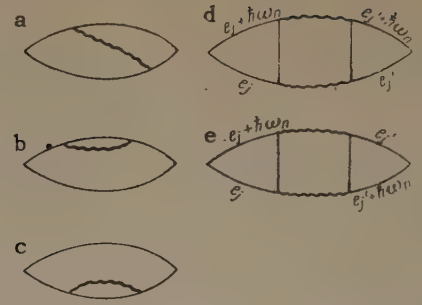


FIG. 2

Because of the presence of the Coulomb divergence the interaction must be renormalized. The renormalization reduces to the result^[1,2] that we must calculate the graphs shown in Fig. 2 in place of those shown in Figs. 1b and c. The thick wavy line satisfies the equation shown in Fig. 3.

This equation corresponds to the function

$$D_\gamma(\alpha_m) = \{P_\gamma(\alpha_m) + P'_\gamma(\alpha_m) - U_\gamma^{-1}\}^{-1}, \quad (10)$$

$$P_\gamma(\alpha_m) = (2\pi)^{-3} \int d^3 q (n_{q+\gamma/2} - n_{q-\gamma/2}) (\epsilon_{q+\gamma/2} - \epsilon_{q-\gamma/2} - \alpha_m)^{-1}, \quad (11)$$

$$P'_\gamma(\alpha_m) = (2\pi)^{-3} \int d^3 q (n'_{q+\gamma/2} - n'_{q-\gamma/2}) (\epsilon'_{q+\gamma/2} - \epsilon'_{q-\gamma/2} - \alpha_m)^{-1}, \quad (12)$$

where $\alpha_m = 2m\pi T$, $n_q = \exp(-\epsilon_q/T)$; the primed quantities n'_q and ϵ'_q are obtained from the unprimed ones by replacing the electron mass by the ion mass.

4. We proceed to the calculation of the graphs of Fig. 2. We shall consider the case in which the wavelength of the electromagnetic wave is much greater than the maximum impact parameter. Under these conditions, we can neglect spatial dispersion.*

The expressions corresponding to the graphs in Figs. 2a, b, c have the form

$$K_{pp'}^{(a)}(\omega_n) = T \sum_m D_{p-p'}(\alpha_m) T \sum_j G_p(e_j + \hbar\omega_n) G_p(e_j) \times G_{p'}(e_j + \alpha_m + \hbar\omega_n) G_{p'}(e_j + \alpha_m),$$

$$K_{pp'}^{(b)}(\omega_n) = \int \frac{d^3 \gamma}{(2\pi)^3} T \sum_m D_\gamma(\alpha_m) T \sum_j [G_p(e_j + \hbar\omega_n)]^2 \times G_{p+\gamma}(e_j + \alpha_m + \hbar\omega_n) G_p(e_j) (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}'),$$

$$K_{pp'}^{(c)}(\omega_n) = \int \frac{d^3 \gamma}{(2\pi)^3} T \sum_m D_\gamma(\alpha_m) T \sum_j G_p(e_j + \hbar\omega_n) [G_p(e_j)]^2 \times G_{p+\gamma}(e_j + \alpha_m) (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}').$$

Carrying out the summation over j and adding all three expressions, we get

*Account of spatial dispersion may prove to be essential in an analysis of the surface impedance in the case $c/\omega_0 < v/\omega$ (ω_0 is the plasma frequency, v is the thermal velocity).



FIG. 3

$$(2\pi)^{-6} \int d^3p d^3p' \rho_{\mu} \rho_{\nu} [K_{pp'}^{(a)}(\omega_n) + K_{pp'}^{(b)}(\omega_n) + K_{pp'}^{(c)}(\omega_n)] \\ = (2\pi)^{-3} (\hbar\omega_n)^{-2} \int \gamma_{\mu} \gamma_{\nu} d^3\gamma T \sum_m D_{\gamma}(\alpha_m) [P_{\gamma}(\alpha_m + \hbar\omega_n) - P_{\gamma}(\alpha_m)] \quad (13)$$

Here P_{γ} is determined by Eq. (11). In similar fashion, after summation over j and j' , the sum of the graphs represented in Figs. 2d and e leads to the expression

$$(2\pi)^{-6} \int d^3p d^3p' \rho_{\mu} \rho_{\nu} [K_{pp'}^{(d)}(\omega_n) + K_{pp'}^{(e)}(\omega_n)] \\ = (2\pi)^{-3} (\hbar\omega_n)^{-2} \frac{1}{2} \int \gamma_{\mu} \gamma_{\nu} d^3\gamma T \sum_m D_{\gamma}(\alpha_m) D_{\gamma}(\alpha_m + \hbar\omega_n) \\ \times [P_{\gamma}(\alpha_m + \hbar\omega_n) - P_{\gamma}(\alpha_m)] \quad (14)$$

Making use of Eq. (10) [after addition of the expressions (13) and (14)], we get

$$(2\pi)^{-6} \int d^3p d^3p' \rho_{\mu} \rho_{\nu} K_{pp'}^{(2)}(\omega_n) \\ = \int \frac{d^3\gamma}{(2\pi)^3} \frac{\gamma_{\mu} \gamma_{\nu}}{2(\hbar\omega_n)^2} T \sum_m D_{\gamma}(\alpha_m) D_{\gamma}(\alpha_m + \hbar\omega_n) \{P_{\gamma}(\alpha_m + \hbar\omega_n) - P_{\gamma}(\alpha_m)\} \{P_{\gamma}(\alpha_m - \hbar\omega_n) - P_{\gamma}(\alpha_m)\} \quad (15)$$

Summation over m in Eq. (15), and analytic continuation in ω can be carried out as follows. Let $F(\omega)$ be the analytic continuation of the function

$$T \sum_m \varphi(\alpha_m) \psi(\alpha_m + \hbar\omega_n)$$

in the variable ω from discrete points on the imaginary axis ($n > 0$) to the real axis. Further, let the functions $\varphi(\alpha)$ and $\psi(\alpha)$ coincide in the upper half plane with the analytic functions $\varphi^R(\alpha)$ and $\psi^R(\alpha)$, and in the lower half plane with the functions $\varphi^A(\alpha)$ and $\psi^A(\alpha)$, wherein the functions $\varphi^R(\alpha)$ and ψ^R are obtained on the real axis from the functions φ^A and ψ^A by complex conjugation and $\varphi(0) = [\varphi^R(0) + \varphi^A(0)]/2$, $\psi(0) = [\psi^R(0) + \psi^A(0)]/2$. Then the following relation holds:*

$$\text{Im } F(\omega) = \text{sh} \frac{\hbar\omega}{2T} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \frac{\text{Im } \varphi^R(\alpha) \text{Im } \psi^R(\alpha + \hbar\omega)}{\text{sh}(\alpha/2\pi) \text{sh}[(\alpha + \hbar\omega)/2T]} \quad (16)$$

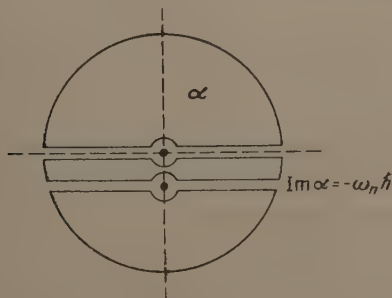


FIG. 4

*sh = sinh.

The integral is taken in the sense of its principal value. The relation (16) follows from the formula

$$T \sum_m \varphi(\alpha_m) \psi(\alpha_m + \hbar\omega_n) \\ = \frac{1}{4\pi i} \int_C \text{cth} \frac{\alpha}{2T} \varphi(\alpha) \psi(\alpha + \hbar\omega_n) d\alpha \\ + T\varphi(0) \psi(\hbar\omega_n) + T\varphi(-\hbar\omega_n) \psi(0). \quad (17)^*$$

The contour C consists of the three closed contours shown in Fig. 4, in which the paths along the large circle are infinitely far away. The integrals over the circle of large radius are equal to zero, so that integration over the contour C reduces to integration over the edges of the two cuts in the integrand functions.

The integrals over the small circles surrounding the points $\alpha = 0$ and $\alpha = -\omega_n$ exactly cancel out the last two terms on the right side of (17).

Thus,

$$T \sum_m \varphi(\alpha_m) \psi(\alpha_m + \hbar\omega_n) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \text{cth} \frac{\alpha}{2T} \{[\varphi^R(\alpha) - \varphi^A(\alpha)] \psi^R(\alpha + \hbar\omega_n) + [\psi^R(\alpha) - \psi^A(\alpha)] \varphi^A(\alpha - \hbar\omega_n)\} d\alpha,$$

where the integral is taken in the sense of its principal value. Analytic continuation now leads to the replacement of ω_n by ω , and Eq. (16) is obtained without difficulty.

Making use of Eq. (16) and the fact that $K_{pp'}(0)$ is a real quantity, we obtain (in the approximation of interest to us) an expression for the real part of the conductivity:

$$\text{Re } \sigma(\omega) = \frac{e^2 (2\pi)^{-4}}{3m^2 \omega^3} \text{sh} \frac{\hbar\omega}{2T} \int_{-\infty}^{\infty} \gamma^2 d^3\gamma \int_{-\infty}^{\infty} d\alpha \left[\text{sh} \frac{\alpha}{2T} \text{sh} \frac{(\alpha + \hbar\omega)}{2T} \right]^{-1} \\ \times \{ \text{Im} [D_{\gamma}^R(\alpha) P_{\gamma}^R(\alpha)] \text{Im} [D_{\gamma}^R(\alpha + \hbar\omega) P_{\gamma}^R(\alpha + \hbar\omega)] \\ - \text{Im} [D_{\gamma}^R(\alpha + \hbar\omega)] \text{Im} [D_{\gamma}^R(\alpha) P_{\gamma}^R(\alpha) P_{\gamma}^R(\alpha)] \}. \quad (18)$$

Here $\sigma(\omega) = \sigma_{XX}(\omega)$. We have made use of the isotropy of the problem and replaced γ_X^2 by $\gamma^2/3$ under the integral. The index R denotes the functions obtained from (10) – (12) by replacement of α_m by $\alpha + i\delta$; $\delta \rightarrow +0$.

Let us introduce the dimensionless quantities

$$x = \left(\frac{m}{T}\right)^{1/2} \frac{\alpha}{\hbar\gamma}, \quad \tau = \frac{\gamma}{\kappa}, \quad \eta^2 = \frac{\hbar^2 \kappa^2}{2mT}, \quad \rho^2 = \frac{m}{M}, \quad \omega^* = \frac{\omega}{\omega_0},$$

*cth = coth.

where $\kappa = (4\pi n_0 e^2/T)^{1/2}$ is the reciprocal of the Debye radius, n_0 is the concentration of electrons (ions), $\omega_0 = (4\pi n_0 e^2/m)^{1/2}$ is the electron plasma frequency.

After reduction of similar terms in the curly brackets and integration over the angles, we write the formula (18) in the following form:

$$\Gamma_{\pm} \sigma(\omega) = \frac{n_0 e^2}{m \omega^2} \frac{2T}{\hbar \omega} \operatorname{sh} \frac{\hbar \omega}{2T} \cdot \frac{2}{3} \pi n_0 \left(\frac{e^2}{T} \right)^2 \left(\frac{8T}{\pi m} \right)^{1/2} (Q + Q' + Q''); \quad (19)$$

$$Q = (2\pi)^{-1/2} \frac{1}{\rho} \int_0^\infty \tau^3 d\tau \int_{-\infty}^\infty dx \left| D_\tau(x) D_\tau \left(x + \frac{\omega^*}{\tau} \right) \right|^2 \Phi(x, \tau) \times \exp \{ -\eta^2 \tau^2 (1 + \rho^2)/4 - x^2/2\rho^2 - (x + \omega^*/\tau)^2/2 \}, \quad (20)$$

$$Q' = -(2\pi)^{-1/2} \int_0^\infty \tau^3 d\tau \int_{-\infty}^\infty dx \left| D_\tau(x) D_\tau \left(x + \frac{\omega^*}{\tau} \right) \right|^2 F'(x, \tau) \times \exp \{ -\eta^2 \tau^2/2 - x^2/2 - (x + \omega^*/\tau)^2/2 \}, \quad (21)$$

$$Q'' = -(2\pi)^{-1/2} \frac{1}{\rho^2} \int_0^\infty \tau^3 d\tau \int_{-\infty}^\infty dx \left| D_\tau(x) D_\tau \left(x + \frac{\omega^*}{\tau} \right) \right|^2 F(x, \tau) \times \exp \{ -\eta^2 \tau^2 \rho^2/2 - x^2/2\rho^2 - (x + \omega^*/\tau)^2/2\rho^2 \}. \quad (22)$$

Here the following nondimensional functions have been introduced:

$$\begin{aligned} |D_\tau(x)|^2 &= [\tau^2 + \varphi(x) + \varphi(x/\rho)]^2 + [\psi(x) + \psi(x/\rho)]^2, \\ \Phi(x, \tau) &= \{\tau^2 + \varphi[\rho^{-1}(x + \omega^*/\tau)] + \varphi(x)\}^2 \\ &\quad + \{\psi[\rho^{-1}(x + \omega^*/\tau)] + \psi(x)\}^2, \\ F(x, \tau) &= \varphi(x) [\varphi(x + \omega^*/\tau) - \varphi(x)] + \psi(x) [\psi(x + \omega^*/\tau) - \psi(x)], \\ F'(x, \tau) &= \varphi(x/\rho) \{\varphi[\rho^{-1}(x + \omega^*/\tau)] - \varphi(x/\rho)\} \\ &\quad + \psi(x/\rho) \{\psi[\rho^{-1}(x + \omega^*/\tau)] - \psi(x/\rho)\}, \\ \varphi(x) &= (2\pi)^{-1/2} \int_{-\infty}^\infty e^{-y^2/2} \frac{y dy}{y-x} = 1 - x e^{-x^2/2} \int_0^x e^{t^2/2} dt, \\ \psi(x) &= (\pi/2)^{1/2} x e^{-x^2/2}. \end{aligned} \quad (23)$$

The functions $\varphi(x)$ and $\psi(x)$ are obtained by reduction of the real and imaginary parts, respectively, of the function PR to nondimensional form. In this case we have used the fact that under the assumptions made in the work the quantity η is a small parameter.

We note that, in spite of the smallness of η , it is impossible for it to approach zero in the exponent of the term Q , for then a logarithmic divergence arises in the integral over τ at the upper limit. We can, however assume $\eta = 0$ in the terms Q' and Q'' .

5. Further simplifications are possible if we take it into account that $m/M \equiv \rho^2 \ll 1$ in the case of the electron-ion plasma under discussion. It is not difficult to prove that this makes it possible to neglect the terms Q' and Q'' . In addition,

we can write the quantity $\Phi(x, \tau) |D_\tau(x + \omega^*/\tau)|^2$ in the term Q in the form

$$(\tau^2 + 1)^2 \{ [\tau^2 + \varphi(\omega^*/\tau)]^2 + [\psi(\omega^*/\tau)]^2 \}^{-1}.$$

Taking it into account that values of x of order ρ are significant in the integral, it is easy to see that when $\omega^*/\tau \gg \rho$ this substitution is exact. On the other hand, for $\omega^*/\tau \ll 1$, both expressions are the same and are equal to unity.

In this way, we get (after substitution of the variable $x/\rho = z$ and simplifications associated with the smallness of ρ)

$$Q = \left(\frac{2}{\pi} \right)^{1/2} \int_0^\infty \tau^3 d\tau e^{-\eta^2 \tau^2/4} \frac{(\tau^2 + 1)^2 \exp(-\omega^{*2}/2\tau^2)}{[\tau^2 + \varphi(\omega^*, \tau)]^2 + [\psi(\omega^*, \tau)]^2} \times \int_0^\infty e^{-z^2/2} \{ [\tau^2 + 1 + \varphi(z)]^2 + [\psi(z)]^2 \}^{-1} dz. \quad (24)$$

We note that although the ratio of the masses of the electron and ion does not appear in Eq. (24), it would have been incorrect to consider the ions to be at rest from the very beginning, since they in fact take part in the screening of the electronic interaction. Were we to assume that the ions are fixed, then we would obtain an expression [in place of Eq. (24)] which does not contain the functions $\varphi(z)$ and $\psi(z)$ in the denominator under the integral in z .

6. Let us consider some limiting cases. We shall write out the final results for the effective collision frequency, which is connected with the conductivity by the relation

$$\operatorname{Re} \sigma(\omega) = (n_0 e^2 / m \omega^2) \nu(\omega).$$

In accord with Eq. (19)

$$\nu(\omega) = \nu_0 \frac{2T}{\hbar \omega} \operatorname{sh} \frac{\hbar \omega}{2T} Q(\omega),$$

where $\nu_0 = 2/3 \pi (e^2/T)^2 n_0 (8T/\pi m)^{1/2}$ and Q is determined by Eq. (24).

1) The frequency of the wave is much larger than the electron plasma frequency ($\omega^* \gg 1$). In this case, the values $\tau \sim \omega^* \gg 1$ are important in the integral over τ . Therefore we can write Eq. (24) in the form

$$Q = \int_0^\infty \frac{d\tau}{\tau} \exp \left[-\eta^2 \frac{\tau^2}{4} - \frac{\omega^{*2}}{2\tau^2} \right] = K_0 \left(\frac{\eta \omega^*}{\sqrt{2}} \right).$$

Here K_0 is the Bessel function of imaginary argument. Noting that $\eta \omega^* / \sqrt{2} = \hbar \omega / 2T$, we get for the collision frequency in the case under discussion:

$$\nu = \nu_0 \frac{2T}{\hbar \omega} \operatorname{sh} \frac{\hbar \omega}{2T} K_0 \left(\frac{\hbar \omega}{2T} \right), \quad \omega \gg \omega_0. \quad (25)$$

Within the framework of this limiting case, we can consider two frequency regions. Formula (25)

yields

$$\nu = \nu_0 \ln (4T A / \hbar \omega), \quad T \gg \hbar \omega \gg \hbar \omega_0; \quad (26)$$

$$\nu = \nu_0 \sqrt{\pi} (T / \hbar \omega)^{1/2}, \quad \hbar \omega \gg T \quad (27)$$

(here $A = e^{-\gamma} = 0.56$, γ is Euler's constant). We note that in the last case the effective collision frequency does not depend on the temperature.

The absorption of electromagnetic radiation for $\omega \gg \omega_0$ was considered in application to semiconductors.^[7] If the expression obtained in the papers cited is supplemented by consideration of induced radiation, it becomes equivalent to (25).

2) The frequency of the wave is much less than the plasma frequency, $\omega^* \ll 1$. In this case, we can set $\omega^* = 0$ in Eq. (24). It is convenient to make the substitution $(\eta \tau)^2 = x$. We have

$$Q = (2\pi)^{-1/2} \int_0^\infty e^{-x^{1/2}} dz \int_0^\infty e^{-x^{1/2}} \frac{x dx}{(x + \eta^2 [1 + \Phi(z)]^2 + \eta^4 \{\Psi(z)\}^2)} \quad (28)$$

As $\eta \rightarrow 0$, this expression diverges logarithmically. To eliminate the logarithmic term we carry out integration over x by parts. We can then set $\eta = 0$ in the integral term. We then obtain*

$$Q = \ln (B_0 / \eta);$$

$$\ln B_0 = \ln 2 - \frac{\gamma}{2} - (2\pi)^{-1/2} \int_0^\infty e^{-z^{1/2}} f(z) dz,$$

$$f(z) = \frac{1}{2} \ln \{ [1 + \Phi(z)]^2 + [\Psi(z)]^2 \}$$

$$- \frac{1 + \Phi(z)}{\Psi(z)} \left\{ \arctg \frac{1 + \Phi(z)}{\Psi(z)} - \frac{\pi}{2} \right\}.$$

Numerical calculation yields

$$\ln B_0 = -0.29, \quad B_0 = 0.75.$$

Thus,

$$\nu = \nu_0 \ln [0.75 (2mT)^{1/2} / \hbar \kappa]. \quad (29)$$

* $\arctg = \tan^{-1}$.

This result could, of course, have been obtained from the kinetic equation for the high temperature plasma.^[2] We note that the numerical factor under the logarithm sign is approximately one third the value used by Spitzer.^[5]

3) In the intermediate case ($\omega^* \sim 1$) numerical calculation is necessary, which it is convenient to carry out for the difference $Q(\omega) - Q(0)$, since in this case the logarithmic singularity at the upper limit is eliminated as $\eta \rightarrow 0$. The result can be written in the following form:

$$\nu = \nu_0 \ln (B_\omega \sqrt{2mT} / \hbar \kappa). \quad (30)$$

Calculation gives, for $\omega = 0.5\omega_0$, ω_0 and $2\omega_0$, the corresponding values of $\ln B_\omega$: -0.29 , -0.30 , -0.48 . As is seen, the value of the factor under the logarithm sign, obtained for $\omega \ll \omega_0$ is shown to be valid in a much wider range of frequencies than one could have expected.

¹A. I. Larkin, JETP 37, 264 (1959), Soviet Phys. JETP 10, 186 (1960).

²O. V. Konstantinov and V. I. Perel', JETP 39, 861 (1960), Soviet Phys. JETP 12, 597 (1961).

³L. D. Landau, JETP 7, 203 (1937).

⁴V. P. Silin, JETP 38, 1771 (1960), Soviet Phys. JETP 11, 1277 (1960).

⁵L. Spitzer, The Physics of Fully Ionized Gases (Russian translation, IIL, 1957).

⁶Abrikosov, Gor'kov, and Dzyaloshinskii, JETP 36, 900 (1959), Soviet Phys. JETP 9, 636 (1959).

⁷R. Wolf, Proc. Phys. Soc. (London) A67, 74 (1954); Fan, Spitzer and Collins, Phys. Rev. 101, 566 (1956).

Translated by R. T. Beyer

EFFECT OF SUPERFLUIDITY OF ATOMIC NUCLEI ON THE STRIPPING AND PICKUP REACTIONS

B. L. BIRBRAIR

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 8, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 894-897 (September, 1961)

Formulas for stripping and pickup reactions in the forward hemisphere are derived by taking superfluidity of atomic nuclei into account. The nuclear model proposed can be verified by comparing the results with the experimental data.

WHEN the energy of the incoming particles is greater than the Coulomb barrier of the target nucleus, the main contribution to the stripping and pickup reactions is made by the direct and exchange effects,^[1] provided there are no isolated resonances of the compound nucleus in the energy region under consideration. This condition is always satisfied in nuclei with $A \sim 40$ at incoming-particle energies on the order of 10 Mev. The direct reaction mechanism^[2] is due to interaction between the nucleons initially contained in the incoming particle, and describes the cross section in the forward hemisphere (in the c.m.s.). Among the exchange effects are knock-out and heavy-particle stripping,^[3] the latter describing the cross section in the rear hemisphere, while knock-out is estimated^[1,4] to make a negligibly small contribution to the cross section of the direct mechanism. In an examination of the cross section in the forward hemisphere it is therefore sufficient to account for the direct reaction mechanism.

Let us consider for simplicity the reactions (d, p), (d, n) and their inverses. Within the framework of the method of distorted waves,^[5,6] usually used for a description of the direct interactions, the amplitude of the reaction is

$$S_{i \rightarrow k} = \langle \Phi_f | \int \varphi_k^*(x_f) V(x_c - x_f) \psi_K^+(x_c x_f) dx_f | \Phi_i \rangle. \quad (1)$$

where Φ_f and Φ_i are respectively the wave functions of the product and target nuclei, $\varphi_k^*(x_f)$ and $\psi_K^+(x_c x_f)$ are the wave function of the outgoing nucleon and of the incoming deuteron with corresponding asymptotic approximations, $\hbar K$ is the deuteron momentum, and $\hbar k$ is the nucleon momentum. Expanding the integral in (1) in terms of the eigenfunctions $\varphi_{jm}(x_c)$ of the captured nucleon in the self-consistent field of the nucleus, we obtain

$$\int \varphi_k^*(x_f) V(x_c - x_f) \psi_K^+(x_c x_f) dx_f = \sum_{jm} \varphi_{jm}(x_c) F(Kkj; \mu_D \mu_f m), \quad (2)$$

where μ_D and μ_f are spin projections of the deuteron and the nucleon and $F(Kkj; \mu_D \mu_f m)$ are the matrix elements that determine the angular distribution of the reaction products. They can be calculated by choosing the optical potentials of the deuteron and nucleon and the interaction potential $V(x_c - x_f)$.^[6] The problem then reduces to a calculation of the matrix elements $\langle \Phi_f | \varphi_{jm} | \Phi_i \rangle$.

Following Belyaev^[7,8] we describe the states Φ_f and Φ_i in the independent quasiparticle approximation, the state operators being related to the nucleon operators by the Bogolyubov canonical transformation

$$a_{jm} = u_j \alpha_{jm} + (-1)^{j-m} v_j \alpha_{j-m}^+, \quad u_j^2 = \frac{1}{2} \left[1 + \frac{\epsilon_j - \lambda}{E_j} \right], \\ v_j^2 = \frac{1}{2} \left[1 - \frac{\epsilon_j - \lambda}{E_j} \right], \quad E_j = \sqrt{(\epsilon_j - \lambda)^2 + \Delta_j^2}, \quad (3)$$

where ϵ_j is the energy of independent motion of the nucleons in the field of the nucleus, λ is the chemical potential (which coincides with the Fermi level when $\Delta = 0$), and E_j is the energy of the quasiparticle. The fundamental and low-lying states of the nuclei with odd A contain one quasiparticle, and the number of quasiparticles in the ground and excited states of even-even nuclei is respectively zero and two (collective excitations of the type considered by Belyaev^[8] are not taken into account).

We start with an examination of the stripping reaction on a target nucleus with odd A . If the even-even product nucleus is in the ground state, we readily obtain with the aid of (1) - (3)

$$S_{Kj \rightarrow k0} = (-1)^{j-m} v_j F(Kkj; \mu_D \mu_f m), \\ \frac{d\sigma(Kj \rightarrow k0)}{d\Omega} = \frac{m_f^2 v_f}{4\pi^2 \hbar^4 v_D} \frac{v_j^2}{3(2j+1)} \sum_{m \mu_D \mu_f} |F(Kkj; \mu_D \mu_f m)|^2. \quad (4)$$

In the case when the product nucleus is in the excited state

$$S_{Kj \rightarrow K'j_1j_2J} = u_{j_1} C_{j_1m_1m}^{JM} F(KK'j; \mu_D \mu_f m_1) \delta_{j_2j} - u_{j_2} C_{j_2m_2m}^{JM} F(KK'j; \mu_D \mu_f m_2) \delta_{j_1j}, \quad (5)$$

i.e., the state of the quasiparticle in the target-nucleus is not changed by the reaction. In particular, when $j_1 = j_2 = j$, i.e., when the states of the quasiparticles in the product nucleus coincide with the state of the quasiparticle in the target, we have

$$S_{Kj \rightarrow K'j^2J} = \sqrt{2} u_j C_{j m_1 m}^{JM} F(KK'j; \mu_D \mu_f m_1),$$

$$\frac{d\sigma(Kj \rightarrow K'j^2J)}{d\Omega} = \frac{m_f^2 v_f^2}{4\pi^2 \hbar^4 v_D} \frac{2(2J+1) u_j^2}{3(2j+1)^2} \sum_{m \mu_D \mu_f} |F(KK'j; \mu_D \mu_f m)|^2, \quad (6)$$

where m_f and v_f are the mass and velocity of the outgoing nucleon, and v_D is the velocity of the incoming deuteron. A comparison of (4) and (6) shows that at sufficiently high incident-particle energies the angular distributions coincide and the limit of the ratio of the corresponding intensities is

$$\lim_{v_f \rightarrow v_f} \frac{d\sigma(Kj \rightarrow K'j^2J)}{d\sigma(Kj \rightarrow K0)} = \frac{2(2J+1) u_j^2}{2j+1} \frac{v_j^2}{v_f^2}. \quad (7)$$

For pickup reactions on odd targets we have

$$S_{Kj \rightarrow K0} = u_j F^*(Kkj; \mu_D \mu_f m),$$

$$\frac{d\sigma(kj \rightarrow K0)}{d\Omega} = \frac{m_D^2 v_D^2}{4\pi^2 \hbar^4 v_f} \frac{u_j^2}{2(2j+1)} \sum_{m \mu_D \mu_f} |F(Kkj; \mu_D \mu_f m)|^2, \quad (8)$$

$$S_{Kj \rightarrow K'j^2J} = (-1)^{j-m_j} \sqrt{2} v_j C_{j m_1 m}^{JM} F^*(K'kj; \mu_D \mu_f m_1),$$

$$\frac{d\sigma(kj \rightarrow K'j^2J)}{d\Omega} = \frac{m_D^2 v_D^2}{4\pi^2 \hbar^4 v_f} \frac{(2J+1) v_j^2}{(2j+1)^2} \sum_{m \mu_D \mu_f} |F(K'kj; \mu_D \mu_f m)|^2, \quad (9)$$

$$\lim_{v_D \rightarrow v_D} \frac{d\sigma(kj \rightarrow K'j^2J)}{d\sigma(kj \rightarrow K0)} = \frac{2(2J+1) v_j^2}{2j+1} \frac{v_j^2}{u_j^2}. \quad (10)$$

Let us consider now the stripping and pickup reactions on even-even targets. Since the corresponding cross sections are connected with (8) and (4) by the principle of detailed balance, we have in the case of stripping

$$\frac{d\sigma(K0 \rightarrow kj)}{d\Omega} = \frac{m_f^2 v_f^2}{4\pi^2 \hbar^4 v_D} \frac{u_j^2}{3} \sum_{m \mu_D \mu_f} |F(Kkj; \mu_D \mu_f m)|^2, \quad (11)$$

and in the case of pickup

$$\frac{d\sigma(k0 \rightarrow Kj)}{d\Omega} = \frac{m_D^2 v_D^2}{4\pi^2 \hbar^4 v_f} \frac{v_j^2}{2} \sum_{m \mu_D \mu_f} |F(Kkj; \mu_D \mu_f m)|^2. \quad (12)$$

It is interesting to compare (11) and (12) with the predictions of the single-particle model. Since we have when $\Delta = 0$

$$u_j^2 = \begin{cases} 1, & \epsilon_j > \lambda \\ 0, & \epsilon_j < \lambda \end{cases}, \quad v_j^2 = \begin{cases} 0, & \epsilon_j > \lambda \\ 1, & \epsilon_j < \lambda \end{cases}$$

only nucleon levels can be excited in stripping reactions, according to the single-particle model, and only hole levels can be excited in pickup reactions. An account of superfluidity, as can be seen from (11) and (12), leads to a nonvanishing probability of excitation of hole levels in stripping reactions and nucleon levels in pickup reactions. We can, however, assume here that levels with $\epsilon_j > \lambda$ are excited in stripping reactions and levels with $\epsilon_j < \lambda$ are excited in pickup reactions.

The derived formulas (4) – (12) are applicable to all direct reactions in which the number of nucleons in the nucleus changes by one. They can therefore be verified experimentally by using any of the foregoing reactions. The influence of the superfluidity of atomic nuclei on electromagnetic transitions and on beta decay was already pointed out by Urin^[9] and Grin',^[10]. However, since the number of quasiparticles in the initial and final states in these cases is either the same or differs by two, the corresponding matrix elements contain combinations of the canonical transformation such as $u_{j_1} u_{j_2} \pm v_{j_1} v_{j_2}$ or $u_{j_1} v_{j_2} \pm v_{j_1} u_{j_2}$. In the case of stripping and pickup, however, the number of quasiparticles differs by one, and consequently the matrix elements contain only one coefficient [u_j if an increase (decrease) in number of nucleons is accompanied by an increase (decrease) in the number of quasiparticles, and v_j in the opposite case). The reactions considered can therefore be used to check the given model of the nucleus.

It is of interest to set up the following experiments: a) Direct determination of u_j^2 and v_j^2 by measuring the relative intensity of the transitions to the ground and excited states of even-even nuclei in reactions on targets with odd A. In this case the dt or He³ α pickup reactions are more convenient to use than the pd reaction, for the exothermal nature of the former makes it easier to satisfy the condition $v' \rightarrow v$, needed to cancel out the matrix elements $F(Kkj; \mu_D \mu_f m)$. b) Determination of the dependence of u_j^2 and v_j^2 on the number of external nucleons, by measuring the intensities of excitation of the states of odd nuclei with given l and j in reactions on groups of even-even isotopes, corresponding to the filling of the same shell.

The author is grateful to S. T. Belyaev, L. A. Sliv, and the staff of the Nuclear Physics Institute of the Moscow State University for a discussion of this work.

¹M. A. Nagarajan and M. K. Banerjee, Nucl. Phys. 17, 341 (1960).

²S. T. Butler, Nuclear Stripping Reactions, (Russ. Transl.), IIL, 1960.

³F. E. Owen and L. Madansky, Phys. Rev. **105**, 1766 (1957).

⁴G. Fulton and G. E. Owen, Phys. Rev. **108**, 787 (1957).

⁵G. E. Brown, Revs. Modern Phys. **31**, 893 (1959).

⁶W. Tobocman, Phys. Rev. **115**, 98 (1959).

⁷S. T. Belyaev, Mat. Fys. Medd. Dan. Vid. Selsk. **31**, No. 11, 1959.

⁸S. T. Belyaev, JETP **39**, 1387 (1960), Soviet Phys. JETP **12**, 968 (1961).

⁹M. G. Urin, JETP **38**, 1852 (1960), Soviet Phys. JETP **11**, 1331 (1960).

¹⁰Yu. T. Grin', JETP **39**, 138 (1960), Soviet Phys. JETP **12**, 100 (1961).

Translated by J. G. Adashko
154

ON THE NATURE OF COLLECTIVE LEVELS IN NONSPHERICAL NUCLEI

D. F. ZARETSKII and M. G. URIN

Moscow Engineering Physics Institute

Submitted to JETP editor, April 10, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 898-906 (1961)

A microscopic interpretation of the so-called β - and γ -vibrational levels in nonspherical nuclei is presented. By means of the method of approximate second quantization an equation is derived for determining the frequencies of these levels. Electromagnetic transition probabilities for these levels are computed. The theory is compared with experiment.

1. INTRODUCTION

At present in addition to rotational levels other levels of a collective nature have also been observed experimentally within the domain of strongly deformed nuclei. In particular, among such levels are included the so-called β - and γ -vibrational levels (excitations of the quadrupole type). The β -vibration levels correspond to internal excitations with a component of the angular momentum along the nuclear axis $K = 0$, while the γ -vibrational levels have this component given by $K = 2$. The energies of such levels lie in the range from 0.5 to 1.2 Mev, while the reduced probabilities of electric quadrupole transitions between these levels and the ground state are several times greater than the values obtained from a single particle estimate.

It is generally assumed^[1] that these levels correspond to the excitation of oscillations of the nuclear surface about its equilibrium shape. An estimate of the frequencies of such oscillations made on the basis of the hydrodynamical model yields values $\hbar\omega \sim 1.5 - 2$ Mev which are considerably greater than the observed values. Moreover, it is not possible within the framework of the hydrodynamical model to take into account effects associated with the shell structure of the nucleus. On the other hand, a calculation within the framework of the independent particle model^[2] leads to frequencies of the corresponding oscillations of the order of magnitude of the energy gap between the shells ($\hbar\omega_0 \sim 5 - 6$ Mev).

Within recent years a theory of collective excitations in spherical nuclei has been developed.^[3-5] This theory is based on introducing the quadrupole-quadrupole interaction in addition to the residual interaction between nucleons which leads to pair correlation. The existence of a residual quadrupole-quadrupole interaction follows from experi-

ment. This interaction manifests itself, for example, by the fact that the quadrupole moments of odd nearly-magic nuclei turn out to be greater than those predicted by the single-particle model. For the same reason the probabilities of E2 transitions in these nuclei are enhanced in comparison with the single-particle values.

The residual quadrupole-quadrupole interaction leads to the formation of a bound state of the quasiparticles. It should be expected that a similar effect must also exist in deformed nuclei. The energy required for the creation of a pair of quasiparticles in a deformed nucleus is $\geq 2\Delta$, where Δ is a constant characterizing the energy of the pair correlation. The quadrupole-quadrupole interaction leads to the formation of a bound state of the quasiparticles the energy of which is $< 2\Delta$.

An investigation of collective excitations in a Fermi system can be carried out in two ways: 1) by the method of the two-particle Green's function^[6] and 2) by the method of approximate second quantization.^[7] The collective nature of the lowest 2^+ levels in spherical nuclei has been investigated in Belyaev's paper^[4] by the first method and by Marumori^[3] and Baranger^[5] by the second method.

In this paper we investigate collective excitations of the quadrupole type in deformed nuclei. The investigation is carried out with the aid of the method of approximate second quantization. An equation is obtained from which the frequencies of β - and γ -excitations can be determined. Wavefunctions are obtained for excited states of the type mentioned. Reduced probabilities for E2 and E0 transitions from these states to the ground state are found.

2. THE METHOD OF INVESTIGATION.
EXCITATION SPECTRUM, WAVE FUNCTIONS.

We assume that the component of the angular momentum of the nucleons along the nuclear axis

is a good integral of the motion. We shall write the Hamiltonian in the system of coordinates fixed with respect to the nuclear axes in the form

$$H = H_0 + H_Q, \quad (1)$$

where H_0 is the reduced Bardeen Hamiltonian for nucleons moving in an axially-symmetric deformed potential, while

$$H_Q = -\frac{1}{2} \kappa \sum_{\mu} \sum_{(\lambda)} (q_{2\mu})_{\lambda\lambda'} (q_{2\mu}^*)_{\lambda_1\lambda'_1} a_{\lambda}^{\dagger} a_{\lambda'} a_{\lambda_1}^{\dagger} a_{\lambda'_1}, \quad (2)$$

$q_{2\mu}$ are quadrupole moment operators; $\kappa > 0$ is the quadrupole-quadrupole interaction constant; $a_{\lambda}^{\dagger}$, a_{λ} are the creation and annihilation operators for a particle in the state λ ; λ is a set of quantum numbers for single nucleon states in the given potential. We restrict ourselves to consideration of nucleons of one kind.

The Hamiltonian H_0 can be diagonalized by the method of canonical transformation due to Bogolyubov.^[7] By neglecting the interaction between the quasiparticles we can represent H_0 in the form

$$H_0 = \sum_{\lambda} E_{\lambda} (\alpha_{\lambda}^{\dagger} \alpha_{\lambda} + \beta_{\lambda}^{\dagger} \beta_{\lambda}), \quad E_{\lambda} = \sqrt{\Delta^2 + \epsilon_{\lambda}^2}, \quad (3)$$

where α_{λ} , β_{λ} ; $\alpha_{\lambda}^{\dagger}$, $\beta_{\lambda}^{\dagger}$ are the annihilation and creation operators for quasiparticles of energy E_{λ} ; ϵ_{λ} are the single particle levels referred to the Fermi surface ϵ_0 . The summation in (3) is carried out over the positive components of the angular momentum. It can be shown^[16] that taking into account the interaction between the quasiparticles in (3) leads to small corrections to the energies of the β -levels.

In future we shall be interested in collective excitations of the boson type characterized by the component of the total angular momentum along the nuclear axis $K = 0; 2$. We write $H_Q = H'_Q + H''_Q$ in terms of the operators α and β with

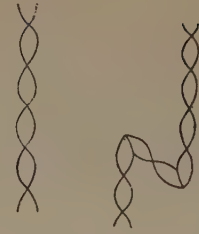
$$H'_Q = -\frac{1}{2} \kappa \sum_{\mu} \sum_{(\lambda)} (q_{2\mu})_{\lambda\lambda'} (q_{2\mu}^*)_{\lambda_1\lambda'_1} (u_{\lambda} v_{\lambda'} + u_{\lambda'} v_{\lambda}) (u_{\lambda_1} v_{\lambda'_1} + u_{\lambda'_1} v_{\lambda_1}) (A_{\lambda\lambda'}^{\dagger} + A_{\lambda'\lambda}) (A_{\lambda_1\lambda'_1}^{\dagger} + A_{\lambda'_1\lambda_1}), \quad (4)$$

Here u_{λ} , v_{λ} are the coefficients of the canonical transformation from the particle operators to the quasiparticle operators:

$$u_{\lambda}^2 = \frac{1}{2} (1 + \epsilon_{\lambda} / E_{\lambda}), \quad v_{\lambda}^2 = \frac{1}{2} (1 - \epsilon_{\lambda} / E_{\lambda}),$$

and $A_{\lambda\lambda'}^{\dagger} = \alpha_{\lambda}^{\dagger} \beta_{\lambda'}^{\dagger}$, $A_{\lambda'\lambda} = \beta_{\lambda} \alpha_{\lambda'}$ are the creation and annihilation operators for pairs of quasiparticles.

In investigating collective excitations we must in our case restrict ourselves to the part of the interaction Hamiltonian H'_Q , and this corresponds to summing graphs of the form shown in the figure.



The neglect of H''_Q in the investigation of collective effects introduces an error $\sim A^{-1/3}$. In the first order approximation with respect to the interaction the operators A , $A^{\dagger}$ can be treated as Bose operators, i.e.,

$$[A_{\lambda\lambda_1}, A_{\lambda'\lambda'_1}^{\dagger}] \approx \delta_{\lambda\lambda'} \delta_{\lambda_1\lambda'_1}. \quad (5)$$

We let

$$H^{red} |K\rangle = E_K |K\rangle, \quad H^{red} |0\rangle = E_0 |0\rangle, \quad (6)$$

$$H^{red} = H_0 + H'_Q,$$

where $|0\rangle$ is the wave function of the ground state, $|K\rangle$ are the wave functions of the excited states of the boson type, characterized by the energy E_K and by the component of angular momentum along the nuclear axis K .

In order to obtain the excitation energy ω_K we write the equations of motion for the operators A , $A^{\dagger}$. In matrix form these equations appear as follows ($\hbar = 1$)

$$\omega_K \langle K | A_{\lambda\lambda'}^{\dagger} | 0 \rangle + \langle K | [A_{\lambda\lambda'}^{\dagger}, H^{red}] | 0 \rangle = 0, \quad (7)$$

$$\omega_K \langle K | A_{\lambda'\lambda} | 0 \rangle + \langle K | [A_{\lambda'\lambda}, H^{red}] | 0 \rangle = 0.$$

By utilizing (3) – (6) we obtain the system of integral equations for the amplitudes $\langle K | A_{\lambda\lambda'}^{\dagger} | 0 \rangle$ and $\langle K | A_{\lambda'\lambda} | 0 \rangle$:

$$\begin{aligned} & (\omega_K - E_{\lambda} - E_{\lambda'}) \langle K | A_{\lambda\lambda'}^{\dagger} | 0 \rangle \\ & + \kappa (q_{2K}^*)_{\lambda\lambda'} f_{\lambda\lambda'} \sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'} f_{\lambda\lambda'} \{ \langle K | A_{\lambda\lambda'}^{\dagger} | 0 \rangle \\ & + \langle K | A_{\lambda'\lambda} | 0 \rangle \} = 0, \\ & (\omega_K + E_{\lambda} + E_{\lambda'}) \langle K | A_{\lambda'\lambda} | 0 \rangle \\ & - \kappa (q_{2K}^*)_{\lambda\lambda'} f_{\lambda\lambda'} \sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'} f_{\lambda\lambda'} \{ \langle K | A_{\lambda\lambda'}^{\dagger} | 0 \rangle \\ & + \langle K | A_{\lambda'\lambda} | 0 \rangle \} = 0, \end{aligned} \quad (8)$$

where we have introduced the notation $f_{\lambda\lambda'} = u_{\lambda} v_{\lambda'} + u_{\lambda'} v_{\lambda}$. The condition that the system (8) should have a solution gives an equation for the determination of ω_K :

$$1 = 2\kappa \sum_{\lambda\lambda'} |(q_{2K})_{\lambda\lambda'}|^2 \frac{E_{\lambda} E_{\lambda'} - \epsilon_{\lambda} \epsilon_{\lambda'} + \Delta^2}{2E_{\lambda} E_{\lambda'}} \frac{E_{\lambda} + E_{\lambda'}}{(E_{\lambda} + E_{\lambda'})^2 - \omega_K^2}. \quad (9)$$

Equation (9) has been obtained previously^[8] by the method of two-particle Green's function. As

can be seen from (9) the frequencies of collective excitations depend on the specific form of the self-consistent potential and on the magnitude of the constant κ .

We now proceed to determine the wave functions of the excited states. We introduce the operators $\mathcal{U}_K$, $\mathcal{U}_K^\dagger$, which satisfy the operator equations

$$\omega_K \mathcal{U}_K + [\mathcal{U}_K, H^{\text{red}}] = 0, \quad \omega_K \mathcal{U}_K - [\mathcal{U}_K, H^{\text{red}}] = 0. \quad (10)$$

On the basis of (6) it follows from (10) that

$$\mathcal{U}_K |0\rangle = 0, \quad \mathcal{U}_K^\dagger |0\rangle = |K\rangle. \quad (11)$$

The normalization condition $\langle K | K' \rangle = \delta_{KK'}$ yields

$$[\mathcal{U}_K, \mathcal{U}_{K'}^\dagger] = \delta_{KK'}. \quad (12)$$

The operators $\mathcal{U}$ and $\mathcal{U}^\dagger$ can be expressed in the form of a linear superposition of the operators A and $A^\dagger$:

$$\begin{aligned} \mathcal{U}_K &= \sum_{\lambda\lambda'} (X_{\lambda\lambda'}^K A_{\lambda\lambda'} - Y_{\lambda\lambda'}^K A_{\lambda\lambda'}^\dagger), \\ \mathcal{U}_K^\dagger &= \sum_{\lambda\lambda'} (X_{\lambda\lambda'}^{K*} A_{\lambda\lambda'}^\dagger - Y_{\lambda\lambda'}^{K*} A_{\lambda\lambda'}), \end{aligned} \quad (13)$$

where the coefficients X and Y satisfy the normalization conditions

$$\sum_{\lambda\lambda'} (X_{\lambda\lambda'}^{K'} X_{\lambda\lambda'}^{K*} - Y_{\lambda\lambda'}^{K'} Y_{\lambda\lambda'}^{K*}) = \delta_{KK'}. \quad (14)$$

With the aid of (10) we obtain the equations for the determination of X and Y :

$$\begin{aligned} (\omega_K - E_\lambda - E_{\lambda'}) X_{\lambda\lambda'}^K &+ \kappa (q_{2K}^*)_{\lambda\lambda'} f_{\lambda\lambda'} \sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'} f_{\lambda\lambda'} (X_{\lambda\lambda'}^K + Y_{\lambda\lambda'}^K) = 0, \\ (\omega_K + E_\lambda + E_{\lambda'}) Y_{\lambda\lambda'}^K &- \kappa (q_{2K}^*)_{\lambda\lambda'} f_{\lambda\lambda'} \sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'} f_{\lambda\lambda'} (X_{\lambda\lambda'}^K + Y_{\lambda\lambda'}^K) = 0. \end{aligned} \quad (15)$$

It can be seen from (15) that in the representation in which the matrix elements $(q_{2K})_{\lambda\lambda'}$ are real, the coefficients X and Y are also real.

3. PROBABILITIES OF ELECTROMAGNETIC TRANSITIONS

The reduced probability of transitions of EL type for deformed nuclei has the form^[9]

$$\begin{aligned} B(EL; I_i K_i \rightarrow I_f K_f) &= e_{\text{eff}}^2 \langle K_f | Q(EL, \mu) | K_i \rangle^2 \langle I_i L K_i K_i - K_f | I_i L I_f K_f \rangle^2, \\ &\quad (16) \end{aligned}$$

where I_i , K_i , I_f , K_f are respectively the spins and the components of angular momentum along the nuclear axis for the initial and the final states; $Q(EL, \mu)$ is the operator for the multipole moment in the system of coordinates fixed with respect to the nuclear axes; e_{eff} is the effective charge. The second factor in (16) is a Clebsch-

Gordan coefficient. In the case $K_i = 0$ and $K_f \neq 0$ the magnitude of $B(EL)$ is greater by a factor of two than would follow from (16).

We first obtain the expressions for $B(E2)$. In terms of α and β , the quadrupole moment operator has the form

$$\begin{aligned} Q_{2\mu} &= 2 \sum_{\lambda} (q_{20})_{\lambda\lambda} v_{\lambda}^2 \delta_{\mu 0} + \sum_{\lambda\lambda'} (q_{2\mu})_{\lambda\lambda'} (u_{\lambda} v_{\lambda'} + u_{\lambda'} v_{\lambda}) (\alpha_{\lambda}^\dagger \beta_{\lambda'}^\dagger + \beta_{\lambda} \alpha_{\lambda'}) + \sum_{\lambda\lambda'} (q_{2\mu})_{\lambda\lambda'} (u_{\lambda} u_{\lambda'} - v_{\lambda} v_{\lambda'}) (\alpha_{\lambda}^\dagger \alpha_{\lambda'} + \beta_{\lambda}^\dagger \beta_{\lambda'}) \\ &= Q_0 \delta_{\mu 0} + Q_{2\mu}^{\prime} + Q_{2\mu}^{\prime\prime}. \end{aligned} \quad (17)$$

The operator $Q_{2\mu}^{\prime\prime}$ makes no contribution to the collective effects, and therefore

$$\begin{aligned} \langle K | Q_{2K} | 0 \rangle &\approx \langle K | Q_{2K}^{\prime} | 0 \rangle \\ &= \sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'} f_{\lambda\lambda'} \langle K | (A_{\lambda\lambda'}^\dagger + A_{\lambda\lambda'}) | 0 \rangle. \end{aligned} \quad (18)$$

It can be easily seen from (13) and (15), that

$$\begin{aligned} X_{\lambda\lambda'}^K &= \kappa (q_{2K}^*)_{\lambda\lambda'} f_{\lambda\lambda'} (E_\lambda + E_{\lambda'} - \omega_K)^{-1} \langle K | Q_{2K} | 0 \rangle, \\ Y_{\lambda\lambda'}^K &= \kappa (q_{2K}^*)_{\lambda\lambda'} f_{\lambda\lambda'} (E_\lambda + E_{\lambda'} + \omega_K)^{-1} \langle K | Q_{2K} | 0 \rangle. \end{aligned} \quad (19)$$

By utilizing the normalization condition (14) we obtain

$$\langle K | Q_{2K} | 0 \rangle^2 = \frac{1}{4\kappa^2 \omega_K} \left\{ \sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'}^2 f_{\lambda\lambda'}^2 \frac{E_\lambda + E_{\lambda'}}{[(E_\lambda + E_{\lambda'})^2 - \omega_K^2]^2} \right\}^{-1}. \quad (20)$$

We proceed to evaluate $B(E0)$. As can be seen from (16), $E0$ transitions are possible only between the states $|0\rangle$ and $|K=0\rangle$ (β -excitation). The matrix element of this transition has the form

$$B^{1/2}(E0; 0^+ \rightarrow 0^+) = \sum_{\lambda\lambda'} (r^2)_{\lambda\lambda'} (X_{\lambda\lambda'}^{K=0} + Y_{\lambda\lambda'}^{K=0}). \quad (21)$$

With the aid of (19) we obtain

$$\frac{B(E0; 0^+ \rightarrow 0^+)}{\langle K=0 | Q_{20} | 0 \rangle^2} = \left[2\kappa \sum_{\lambda\lambda'} (r^2)_{\lambda\lambda'} (q_{20})_{\lambda\lambda'} f_{\lambda\lambda'}^2 \frac{E_\lambda + E_{\lambda'}}{(E_\lambda + E_{\lambda'})^2 - \omega_K^2} \right]^2. \quad (22)$$

It follows from (20) and (22) that the probabilities of $E2$ and $E0$ transitions can be evaluated if we know the set of single particle levels, the corresponding frequencies of collective excitations and the value of the constant κ .

4. ANALYSIS OF RESULTS

The formulas derived in the proceeding sections become considerably simplified in the case when the collective levels lie sufficiently low: $\omega_K \ll 2\Delta$ (adiabatic approximation). If the interval $\sim \Delta$ contains a sufficiently large number of single particle levels ($\rho_0 \Delta \gg 1$), then we can use the quasiclassical approximation. Particu-

larly simple and clear results can be obtained in the case of the deformed oscillator potential. This case has been discussed previously.^[8] For example, in the adiabatic approximation the following relation was obtained for the frequency of the β -excitation

$$\omega_{K=0}/2\Delta = \left\{ \frac{3}{2} \left[1 - \kappa' \sum_{\lambda} (q_{20})_{\lambda\lambda}^2 \delta(\varepsilon_{\lambda}) \right] \right\}^{1/2}, \quad (23)$$

where

$$\kappa' = \kappa \left[1 - \kappa \sum_{\lambda \neq \lambda'} (q_{20})_{\lambda\lambda'}^2 \frac{E_{\lambda} E_{\lambda'} - \varepsilon_{\lambda} \varepsilon_{\lambda'} + \Delta^2}{2E_{\lambda} E_{\lambda'} (E_{\lambda} + E_{\lambda'})} \right]^{-1}.$$

In (23) and in subsequent relations the summation over λ is taken over both signs of the component of angular momentum of the nucleons along the nuclear axis. An estimate of the value of the constant κ , and consequently also of κ' is available^[10]:

$$\kappa' \sim \varepsilon_0 / AR^4, \quad (24)$$

where A is the number of nucleons, R is the nuclear radius. By utilizing the obvious estimate

$$\sum_{\lambda} (q_{20})_{\lambda\lambda}^2 \delta(\varepsilon_{\lambda}) \sim \rho_0 R^4, \quad (25)$$

we obtain

$$\kappa' \sum_{\lambda} (q_{20})_{\lambda\lambda}^2 \delta(\varepsilon_{\lambda}) \sim 1.$$

From this it follows that $\omega_{K=0}$ can be appreciably smaller than the quantity 2Δ .

From (23) it follows, in particular, that the method utilized in this paper is applicable if

$$\kappa' \sum_{\lambda} (q_{20})_{\lambda\lambda}^2 \delta(\varepsilon_{\lambda}) < 1.$$

Equation (9) can be rewritten in the form

$$1 = \kappa'' \omega_K^2 \sum_{\lambda\lambda'} |(q_{2K})_{\lambda\lambda'}|^2 \frac{E_{\lambda} E_{\lambda'} - \varepsilon_{\lambda} \varepsilon_{\lambda'} + \Delta^2}{E_{\lambda} E_{\lambda'} (E_{\lambda} + E_{\lambda'})} \frac{1}{(E_{\lambda} + E_{\lambda'})^2 - \omega_K^2}, \quad (26)$$

where

$$\kappa'' = \kappa \left[1 - \kappa \sum_{\lambda\lambda'} |(q_{2K})_{\lambda\lambda'}|^2 \frac{E_{\lambda} E_{\lambda'} - \varepsilon_{\lambda} \varepsilon_{\lambda'} + \Delta^2}{2E_{\lambda} E_{\lambda'} (E_{\lambda} + E_{\lambda'})} \right]^{-1}.$$

It can be easily shown that in the case of a deformation such that $\beta \ll 1$ the sum appearing in the definition of κ'' is independent of K with quasi-classical accuracy. Therefore, in the approximation $\omega_K \ll 2\Delta$ we obtain the ratio of the frequencies of the β - and γ -excitations in the form

$$\frac{\omega_{\beta}^2}{\omega_{\gamma}^2} = \sum_{\lambda\lambda'} |(q_{22})_{\lambda\lambda'}|^2 \mathcal{K}(\varepsilon_{\lambda}, \varepsilon_{\lambda'}) / \sum_{\lambda\lambda'} |(q_{20})_{\lambda\lambda'}|^2 \mathcal{K}(\varepsilon_{\lambda}, \varepsilon_{\lambda'}), \quad (27)$$

where

$$\mathcal{K}(\varepsilon_{\lambda}, \varepsilon_{\lambda'}) = \frac{E_{\lambda} E_{\lambda'} - \varepsilon_{\lambda} \varepsilon_{\lambda'} + \Delta^2}{2E_{\lambda} E_{\lambda'} (E_{\lambda} + E_{\lambda'})^3}. \quad (27')$$

Thus, in the adiabatic approximation the ratio (27) is independent of the constant κ . In the quasi-classical approximation we obtain (cf. Appendix)

$$\omega_{\beta}^2 / \omega_{\gamma}^2 = \sum_{\lambda\lambda'} |(q_{22})_{\lambda\lambda'}|^2 F_{\lambda\lambda'} / \sum_{\lambda\lambda'} |(q_{20})_{\lambda\lambda'}|^2 F_{\lambda\lambda'}, \quad (28)$$

where we have used the notation

$$F_{\lambda\lambda'} = \frac{\delta(\varepsilon_{\lambda})}{(\varepsilon_{\lambda} - \varepsilon_{\lambda'})^2} \left[1 - g \left(\frac{\varepsilon_{\lambda} - \varepsilon_{\lambda'}}{2\Delta} \right) \right],$$

$$g(x) = \frac{\ln(x + \sqrt{1+x^2})}{x \sqrt{1+x^2}}.$$

In the simplified Nilsson model without spin-orbit coupling^[11] we obtain

$$\omega_{\beta}^2 / \omega_{\gamma}^2 = 3 \int_0^1 [1 - g(tx)] \frac{(1 - 3x^2 + 2x^3)}{(tx)^2} dx, \quad t = \frac{2D\varepsilon_0}{\omega_0 \Delta}. \quad (29)$$

where D is the Nilsson parameter, ω_0 is the gap between the shells in the oscillator potential. As $D \rightarrow 0$ the ratio $\omega_{\beta} / \omega_{\gamma} \rightarrow 1$. It can be shown by using (29) that for $D \neq 0$ the ratio $\omega_{\beta} / \omega_{\gamma} < 1$. Thus, in the adiabatic approximation the energies of the β -excitations are lower than the energies of the γ -excitations.

Systematic data on the position of the levels of β - and γ -excitations are available in the region of Th, U, Pu nuclei. For all these nuclei levels corresponding to β -excitations lie below levels corresponding to γ -excitations.^[12]

A numerical solution of (9) was carried out for the case of β -excitations in the range of nuclei mentioned above. The Nilsson model was used as the single particle model. In expression (9) the sum over $\lambda \neq \lambda'$ was included in the renormalization of the constant κ . The calculation was carried out in accordance with the formula

$$1 = \tilde{\kappa} \sum_{\lambda} (q_{20})_{\lambda\lambda}^2 \frac{\Delta^2}{\sqrt{\Delta^2 - \varepsilon_{\lambda}^2}} \frac{1}{(\varepsilon_{\lambda}^2 + \Delta^2 - \omega_{\beta}^2)}. \quad (30)$$

The value of the constant $\tilde{\kappa}$ was determined from the position of the β -level in the Th²³² nucleus. The values of the deformation β were taken from [13]. The quantity Δ was determined from the binding energy of appropriate nuclei. The energy of the Fermi boundary was identified with the position of the single particle level of the odd neutron in the ground state of the preceding odd nucleus.

The results of the calculation and their comparison with experiment are given in the table. The results of the calculation show that in agreement with experiment ω_{β} varies smoothly from one nucleus to another. It seems to us that this fact supports the hypothesis of the collective nature of this level.

Nucleus	β	Δ , Mev	ϵ_0	ω_β , Mev	
				experiment	theory
Th ²³⁰	0.23	0.85	633, $5/2$	0.635	0.47
Th ²³²	0.24	0.85	633, $5/2$	0.740	—
U ²³²	0.26	0.65	633, $5/2$	0.700	0.79
U ²³⁴	0.25	0.80	633, $5/2$	0.830	0.85
U ²³⁶	0.27	0.70	743, $7/2$	0.925	0.96
U ²³⁸	0.27	0.70	631, $1/2$	0.990	0.78
Pu ²³⁸	0.27	0.70	743, $7/2$	0.940	0.96
Pu ²⁴⁰	0.28	0.60	631, $1/2$	0.860	0.81

We now proceed to analyze the formulas for the probabilities of electromagnetic transitions. In the approximation $\omega_K \ll 2\Delta$

$$\langle K | Q_{2K} | 0 \rangle^2 = \frac{1}{2\kappa^2 \omega_K} \left[\sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'}^2 \mathcal{K}(\epsilon_\lambda, \epsilon_{\lambda'}) \right]^{-1}. \quad (31)$$

In (31) transitions over the shell can be neglected with an accuracy $\sim A^{-1/3}$. In the quasiclassical approximation we obtain (cf. Appendix)

$$\langle K | Q_{2K} | 0 \rangle^2 = \frac{1}{2\kappa^2 \omega_K} \left[\sum_{\lambda\lambda'} (q_{2K})_{\lambda\lambda'}^2 F_{\lambda\lambda'} \right]^{-1}. \quad (32)$$

We now discuss the excitation of β -levels. If the single particle diagram is such that in the sum (30) only diagonal matrix elements are important (for example, the axially-symmetric oscillator potential), then

$$\langle K = 0 | Q_{20} | 0 \rangle^2 = \frac{3\Delta^2}{\kappa^2 \omega_\beta} \left[\sum_{\lambda} (q_{20})_{\lambda\lambda}^2 \delta(\epsilon_\lambda) \right]^{-1}. \quad (33)$$

By utilizing the estimate (25) we obtain

$$B_\beta(E2) \sim e_{\text{eff}}^2 \Delta^2 / \kappa \omega_\beta \sim B_{\text{sp}}(E2) \cdot \rho_0 \Delta \cdot 2\Delta / \omega_\beta, \quad (34)$$

where $B_{\text{sp}}(E2) \sim e_{\text{eff}}^2 R^4$ is the reduced probability for the single particle transition. Thus, it follows from (34) that in the adiabatic case there is an increase in the probability of the E2 transition in comparison with the single particle probability by approximately a factor of $A^{1/3}$.

By utilizing (27) and (31) we obtain in the adiabatic approximation

$$B_\gamma(E2; 0^+ \rightarrow 2^+) / B_\beta(E2; 0^+ \rightarrow 2^+) = 2\omega_\gamma / \omega_\beta. \quad (35)$$

From this it follows that in this approximation

$$B_\gamma(E2; 0^+ \rightarrow 2^+) / B_\beta(E2; 0^+ \rightarrow 2^+) > 2. \quad (36)$$

We go on to the discussion of E0 transitions. For $\omega_\beta \ll 2\Delta$ we obtain from (22) in the quasiclassical approximation

$$\frac{B(E0; 0^+ \rightarrow 0^+)}{B_\beta(E2; 0^+ \rightarrow 2^+)} \approx \kappa^2 \left[\sum_{\lambda\lambda'} (r^2)_{\lambda\lambda'} (q_{20})_{\lambda\lambda'} \delta(\epsilon_\lambda) \right]^2. \quad (37)$$

On introducing the particle density near the Fermi surface

$$\rho(\epsilon_0, r) = \sum_{\lambda} \varphi_{\lambda}^*(r) \varphi_{\lambda}(r) \delta(\epsilon_{\lambda}), \quad (38)$$

we obtain

$$\sum_{\lambda\lambda'} (r^2)_{\lambda\lambda'} (q_{20})_{\lambda\lambda'} \delta(\epsilon_{\lambda}) \approx \int \rho(\epsilon_0, r) r^4 Y_{20} dr. \quad (39)$$

The integral can be easily evaluated by assuming that the particle density is constant over the volume of the nucleus. We take the surface of the nucleus to be of the form

$$R(\theta) = R(1 + \beta Y_{20}(\theta)). \quad (40)$$

For the ratio (37) we have

$$B(E0; 0^+ \rightarrow 0^+) / B_\beta(E2; 0^+ \rightarrow 2^+) \approx (\kappa \rho_0 R^7 / V)^2 \beta^2, \quad (41)$$

where V is the volume of the nucleus. By utilizing the estimate (24), and on taking into account that $\rho_0 \sim A/\epsilon_0$, we obtain $\kappa \rho_0 R^7 / V \sim 1$, and consequently

$$B(E0; 0^+ \rightarrow 0^+) / B_\beta(E2; 0^+ \rightarrow 2^+) \sim \beta^2. \quad (42)$$

The estimate (42) is in qualitative agreement with the available experimental data.^[14]

5. CONCLUSION

From the foregoing discussion it follows that the so-called β - and γ -vibrational levels can be interpreted from the microscopic point of view as collective excitations of boson type in a Fermi system of finite dimensions. Relations have been obtained for finding the frequencies of β - and γ -excitations, and also the probabilities of electromagnetic transitions. If the self-consistent potential, and, consequently, the single-particle energy level scheme, are sufficiently well known, then all the aforementioned quantities can be evaluated by means of automatic computers. The magnitude of the quadrupole-quadrupole interaction constant κ for a given domain of deformed nuclei (rare earths or actinides) can be regarded as constant. Therefore, having obtained the value of κ from the known β - and γ -level for one nucleus we can calculate the positions of β - and γ -levels for other nuclei. Knowing the value of the constant κ we can also evaluate the probabilities of electromagnetic transitions for a given group of nuclei.

Unfortunately, at the present time we do not yet have sufficient experimental data pertaining to the aforementioned type of levels. As has been pointed out already, systematic data on the positions of β - and γ -levels are available only for Th, U, Pu nuclei. In the rare earth region there are only a few such data. The low lying 0^+ level has been identified only in a few cases. A search for low-lying 0^+ levels in the region of the rare earths would be of interest. In accordance with the theory, if there exists a low lying 2^+ level corresponding to a γ -excitation, then nearby there should also exist a 0^+ level corresponding to a β -excitation. Measurement of the probabilities of electromagnetic transitions is also of interest since so far there are very few data on this subject and they are not accurate.

In conclusion the authors express their deep gratitude to Yu. T. Grin' who participated in the initial stages of this work.

APPENDIX

We evaluate the sum

$$S \equiv \sum_{\lambda\lambda'} |(q_{2K})_{\lambda\lambda'}|^2 \mathcal{K}(\epsilon_\lambda, \epsilon_{\lambda'}) \quad (\text{A.1})$$

in the quasiclassical approximation. For this we use the method proposed by Migdal.^[15] It can be easily seen that the quantity $\mathcal{K}$ regarded as a function of ϵ_λ has a sharp maximum when the difference $d = \epsilon_\lambda - \epsilon_{\lambda'}$ is held fixed. This maximum has a width of $\sim \Delta$ near the Fermi surface. We denote

$$\int \mathcal{K}(\epsilon_\lambda, \epsilon_\lambda - d) d\epsilon_\lambda = L\left(\frac{d}{2\Delta}\right).$$

Since the width Δ encompasses many levels and $\mathcal{K}$ has the required property near the Fermi surface, we have

$$S \approx \sum_{\lambda\lambda'} |(q_{2K})_{\lambda\lambda'}|^2 L\left(\frac{\epsilon_\lambda - \epsilon_{\lambda'}}{2\Delta}\right) \delta(\epsilon_\lambda). \quad (\text{A.2})$$

It can be easily seen that

$$\begin{aligned} L\left(\frac{d}{2\Delta}\right) &= \frac{1}{4} \int_{-\infty}^{\infty} \frac{d\epsilon}{\sqrt{\Delta^2 + \epsilon^2} \sqrt{\Delta^2 + (\epsilon - d)^2} (\sqrt{\epsilon^2 + \Delta^2} - \sqrt{\Delta^2 + (\epsilon - d)^2})} \\ &\quad - \frac{1}{4d^2} \int_{-\infty}^{\infty} \frac{(\sqrt{\epsilon^2 + \Delta^2} - \sqrt{\Delta^2 + (\epsilon - d)^2})^2 d\epsilon}{\sqrt{\Delta^2 + \epsilon^2} \sqrt{\epsilon - d)^2 + \Delta^2} (\sqrt{\epsilon^2 + \Delta^2} - \sqrt{\Delta^2 + (\epsilon - d)^2})} \\ &= I_1 + I_2. \end{aligned} \quad (\text{A.3})$$

The first integral in (A.3) is equal to

$$I_1 = \frac{1}{(2\Delta)^2} g\left(\frac{d}{2\Delta}\right), \quad (\text{A.4})$$

where

$$g(x) = [x\sqrt{1+x^2}]^{-1} \ln(x + \sqrt{1+x^2}).$$

The integral I_2 can also be easily evaluated and reduces to the expression

$$I_2 = \frac{1}{d^2} \left\{ 1 - \left[1 + \left(\frac{d}{2\Delta}\right)^2 \right] g\left(\frac{d}{2\Delta}\right) \right\}. \quad (\text{A.5})$$

As a result of this we obtain for S with quasi-classical accuracy the expression utilized in (28) and (32).

¹A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, 16 (1953).

²D. Inglis, Phys. Rev. **97**, 701 (1955).

³T. Marumori, Progr. Theoret. Phys. (Kyoto) **24**, 331 (1960).

⁴S. T. Belyaev, JETP **39**, 1387 (1960), Soviet Phys. JETP **12**, 968 (1961).

⁵M. Baranger, Phys. Rev. **120**, 957 (1960).

⁶V. M. Galitskii, JETP **34**, 1011 (1958), Soviet Phys. JETP **7**, 698 (1958).

⁷Bogolyubov, Tolmachev, and Shirkov, Novyi metod v teorii sverkhprovodimosti (A New Method in the Theory of Superconductivity), AN SSSR, 1958.

⁸Yu. T. Grin' and D. F. Zaretskii, Izv. AN SSSR Ser. Fiz. (in press).

⁹Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. **28**, 432 (1956).

¹⁰S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-fiz. Medd. **31**, 11 (1959).

¹¹S. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fiz. Medd. **29**, 16 (1955).

¹²I. Perlman, Proc. Int. Conf. on Nuclear Structure, Kingston, Canada, Toronto Press, 1960, p. 547.

¹³O. Prior, Arkiv Fysik **14**, 451 (1959).

¹⁴J. Rasmussen, Nuclear Phys. **19**, 85 (1960).

¹⁵A. B. Migdal, JETP **37**, 249 (1959), Soviet Phys. JETP **10**, 176 (1960).

¹⁶D. F. Zaretsky and M. G. Urin, Nuclear Phys. (in press).

Translated by G. Volkoff

ON AN UPPER LIMIT ON THE DENSITY OF NEUTRINOS, GRAVITONS, AND BARYONS IN THE UNIVERSE

Ya. B. ZEL'DOVICH and Ya. A. SMORODINSKII

Submitted to JETP editor April 14, 1961

J. Exptl. Phys. (U.S.S.R.) 41, 907-911 (September, 1961)

To estimate the maximum energy density of neutrinos and other weakly interacting particles in the universe we consider the gravitational action of these particles on the expanding universe. It turns out that the energy density does not exceed $2 \times 10^{-28} (\text{g/cm}^3) \cdot c^2 = 2 \times 10^{-7} \text{ erg/cm}^3$; this estimate is much more accurate (lower) than that from nuclear physics experiments. The question of stars with large gravitational mass defects is considered, and in this connection an upper limit on the density of nucleons is given.

Present data do not enable us to exclude the possibility that the density of forms of matter that are not readily observed and the density of nucleons in a state in which they are hard to observe is several times the density of the well known easily observed forms of matter and energy in the form of nucleons in ordinary stars and in interstellar gas.

1. As has been shown by Pontecorvo and one of the writers,^[1] the determination of the cosmic density of neutrinos is an extremely difficult experiment, owing to the very weak interaction of neutrinos with matter. The experiments of Reines and Cowan on the determination of the antineutrino flux from a pile^[2] and the experiments of Davis, who showed that under these conditions reactions requiring neutrinos are absent,^[3] are well known. From these experiments it follows in particular that the flux of cosmic ν and $\bar{\nu}$ is smaller than $W = 10^{12} - 5 \times 10^{11} \text{ cm}^{-2} \text{ sec}^{-1}$ ^[4] for ν and $\bar{\nu}$ with energies of the order of 2-3 Mev. This flux corresponds to a particle density $W/c = \rho_{\nu}, \bar{\nu} = 20 \text{ cm}^{-3}$ and a mass density $E\rho/c^2 \sim 10^{-25} \text{ g/cm}^3$. This quantity is to be compared with the mean density of nucleons in the universe, which is at present believed to be 10^{-29} g/cm^3 (which corresponds to $\sim 10^{-5} \text{ proton/cm}^3$).

Thus direct experiment does not exclude the possibility that the density of mass and energy in the form of neutrinos is 10^4 to 10^5 times the density of rest mass-energy in ordinary forms.

As is remarked by Reines,^[4] and also in ^[1], these estimates actually depend on an assumption about the spectrum of the neutrinos. At high energies the interaction cross section increases as E^2 , and therefore when recomputed for a different neutrino energy the maximum flux varies $\sim 1/E^2$, and the maximum mass-energy density varies as $1/E$. But even for ν and $\bar{\nu}$ with energies $\sim 1 \text{ Bev}$ the maximum mass-energy density is still larger than the density of nucleons.

On the other hand, at small energies, less than the threshold of the reactions in question, the efficiency of registration of neutrinos by present methods falls sharply; the threshold is $\sim 1.8 \text{ Mev}$ for $\bar{\nu} + p \rightarrow n + e^+$ and $\sim 1 \text{ Mev}$ for $\nu + \text{Cl} \rightarrow \text{A} + e^-$. If there are two kinds of neutrinos—"electron neutrinos" and "muon neutrinos"^[5]—then for the second kind the threshold of the reactions

$$\bar{\nu}_{\mu} + p = n + \mu^+, \quad \nu_{\mu} + n = p + \mu$$

is of the order of 100 Mev. Subthreshold neutrinos can be detected only through the ionization caused by elastic collisions, and therefore for them the estimate of the cosmic density consistent with experiment goes up by several more orders of magnitude.

2. A different approach to the setting of an upper limit on the cosmic density of weakly interacting particles is based on consideration of their gravitational action.* The advantage of this approach is that the estimate does not depend on the shape of the spectrum nor on the interaction cross section, and in principle applies also to all possible as yet unknown weakly interacting fields, and also to the density of high-frequency oscillations of the gravitational field (gravitons).†

*The idea of such an approach is already contained in lectures by Einstein.^[6]

†Because of the weakness of the gravitational interaction all of the cross sections of gravitons are so small that there is practically no hope of detecting them (cf., e. g., a calculation by Gandel'man and Pinaev,^[7] according to whom the bremsstrahlung of gravitons is a factor 10^{10} smaller than the production of $\nu - \bar{\nu}$ pairs).

The point of the proposed estimate is that in the present known state of the universe (characterized by a definite value of the Hubble constant h in the equation $v = hr$, where v is the speed and r is the distance) the density ρ of all forms of matter-energy determines the past of the universe. For example, for $\rho \rightarrow 0$ the distance between each pair of distant objects varies linearly with the time and the state of maximum density occurred at the time $T_0 = -1/h$, i.e., approximately 10^{10} years ago.

For nonvanishing ρ the expansion of the universe slows down in the course of time; the average speed of expansion from the time of maximum density to the present time is larger than the experimentally known instantaneous speed, and consequently the past time of maximum density was closer to the present time.*

Let us introduce a critical density value ρ_c . This concept corresponds to the dividing line between the open model and the closed model of the universe; this corresponds physically to qualitatively different predictions about the future of the universe: for $\rho < \rho_c$ the expansion will continue without limit, but for $\rho > \rho_c$ the expansion will be replaced in the course of time by a contraction. The existence of the singularity and the character of the solution for the past are qualitatively independent of the ratio of ρ to ρ_c .

We give here the value of the time interval from the time of the maximum density in the past to the present time as a function of the total density of all forms of matter-energy, in the dimensionless variables $\tau = Th = T/T_0$ as function of $q = \rho/2\rho_c$:

q :	0	0.5	2	5	20	100
τ :	1	0.67	0.47	0.35	0.21	0.10
τ' :	1	0.50	0.33	0.24	0.14	0.07

The quantity τ is here calculated for stationary matter with the pressure p equal to zero. For τ' the calculation is made (in accordance with a remark of L. D. Landau) for the case in which the rest mass can be neglected, the particles are moving with the speed of light, and $p = \epsilon/3$, where ϵ is the energy density. The difference between τ and τ' is not large.

We note that the difference in the behavior of the two models (with $p = 0$ and $p = \epsilon/3$) is due to the decrease of the energy of relativistic particles on adiabatic expansion. This concept can be applied in spite of the fact that the neutrino does not interact with any kind of particles and

does not "do work." In fact, in the course of time there will pass through a given point neutrinos emitted from more and more distant regions, and thus having suffered stronger and stronger red shifts, or Doppler energy losses. The Doppler energy loss is indeed the physical mechanism that brings about the decrease of the mean energy of the neutrinos as the universe expands.

For reference we present the following formulas:*

$$\tau = k^{-2} [(k + k^{-1}) \operatorname{arctg} k - 1], \quad k = (2q - 1)^{1/2};$$

$$\tau' = [1 + (2q)^{1/2}]^{-1}.$$

We take $T_0 = 10^{10}$ years, and the corresponding critical density is $\rho_c = 2 \times 10^{-29}$ g/cm³. Assuming that the time T is not less than the geological age of the earth, which is of the order of 4×10^9 years, we find that $\tau > 0.4$. From this we have $q < 5$ and $\rho < 2q\rho_c = 2 \times 10^{-28}$ g/cm³.

Another possibility for getting an idea of the past of the universe is to consider distant galaxies, since we are actually observing them in a long past state. In the theory of the expanding universe one gets a nonlinear relation of the speed of expansion on the distance, $v = \chi(r)$; the Hubble constant is only the first derivative, $h = \chi'(0)$, and the value of the second derivative $\chi''(0)$ depends on the density. At present $\chi''(0)$ has not been reliably determined, but the upper limit on χ'' also sets a limit on q : $q < 10$; thus two independent estimates of the upper limit are in agreement (cf. [9]).

Finally, we note that one of the methods by which the observed mean density $\bar{\rho} = 10^{-29}$ has been obtained is as follows: from the motion of the stars in a galaxy one determines the mass of the galaxy, and then finds the average density over the entire universe by dividing the average mass M of a galaxy by the average volume V per galaxy in the universe. It is obvious that in this calculation the mass of a galaxy includes the mass of the neutrinos that are inside it. The density of the neutrinos must be the same inside galaxies and in intergalactic space. Therefore still another way of giving an upper limit on the density of neutrinos is to assume that the entire mass of a galaxy, which determines its gravitational field, is the mass of the neutrinos.

We thus have $\rho_\nu \leq M/V_0$, where V_0 is the actual volume of a galaxy. From this we get

$$\rho_\nu \leq (M/V) (V/V_0) = \bar{\rho} V/V_0 \approx \bar{\rho}'/0.1,$$

since on the average the volume of the galaxies makes up about 10 percent of the volume of the universe. Consequently this third estimate also

*The question of the relation between the age of the galaxy and the quantity $1/h$ has also been considered by Hoyle, [8]

* $\operatorname{arctg} = \tan^{-1}$.

agrees in order of magnitude with the preceding ones.

Thus although the density of neutrinos, gravitons, and so on in the universe may indeed exceed the observed mean density of nucleons (10^{-29} g/cm³), it does not do so by more than a factor 10 to 20.

The supposition that the density of energy in unobservable forms is several times that of ordinary matter is surprising, but nevertheless at present it cannot be rejected. We remind the reader that with all its imperfection the gravitational estimate is still more sensitive by a very large factor than the estimate from nuclear physics experiments.

Digressing from the topic of this paper, we note the striking and as yet unexplained close correspondence between the critical value ρ_c and the observed density $\bar{\rho}$ of ordinary matter (2×10^{-29} and 10^{-29} g/cm³). This could mean, for example, that the age of the universe is relatively small. In this case all three models behave about the same, and $\rho \sim \rho_c$. The connection between the Hubble constant and the age τ is given by the formula $\tau = 2/3h$. In the estimates given we have started from the hypothesis of the isotropic model of the universe and have rejected the introduction of a cosmological constant into the equations, and also have rejected the hypothesis of the spontaneous creation of energy.

3. Let us now turn to the question of the density of nucleons in the universe. At first glance, a determination of the mean density of mass in the universe also gives, to whatever accuracy the estimate can be made, the mean density of nucleons, or more exactly, of baryons.* If the mass density of neutrinos is comparable with that of nucleons, then the mass density of nucleons is also less than the accepted value of $\sim 10^{-29}$ g/cm³, which corresponds to a number density of nucleons of $\sim 10^{-5}$ cm⁻³.

We wish here to call attention to the possibility in principle that the number density of nucleons may be much larger than this value. In principle we can imagine a star which after gravitational collapse is in a state in which its gravitational mass defect ΔM is close to M_0 , the sum of the rest masses of the nucleons making up the star. Then its mass will be much smaller than M_0 ,

$$M = M_0 - \Delta M \ll M_0 = Nm_0$$

and the gravitational field produced by such a star

gives a decided underestimate of the number of nucleons in the star, $M/m_0 \ll N$.

The existence of a gravitational mass defect in the general theory of relativity is not open to doubt. For example, when a rarefied cloud of hydrogen atoms is converted into a dense neutron nucleus, a calculation by the classical Newtonian theory of gravitation shows that if the mass of the cloud is large enough the process is energetically favorable; the release of gravitation energy exceeds the loss of energy in the conversion of hydrogen atoms into the heavier neutrons. Such a process must occur with the release of energy to the outside world in the form of light and neutrinos. There is no doubt that when this happens the total mass of the star decreases, in accordance with the energy which it has given up. At the same time the rest mass of the particles composing the star has increased in the transition $H \rightarrow n$. The decrease of the mass of the star (and of its gravitational field at large distances) is a direct result of the gravitational mass defect, that is, of the fact that the mass of a body is less than the sum of the parts composing it by an amount equal to the energy of the gravitational interaction of the parts (divided by c^2).

Thus the existence of the gravitational mass defect is qualitatively obvious, and it can be calculated in an elementary way for weak gravitational fields. In order, however, for the mass defect to be nearly equal to the rest mass, the gravitational potential must be of the order of c^2 , and in this region we must use the exact equations of the general theory of relativity. It is also necessary to know the equation of state, i.e., the dependence of the pressure and energy on the density of nucleons. At present the theory of such a state of a star has not been developed. There are only approximate estimates of the minimum number of nucleons necessary for gravitational collapse to be possible; Landau and Lifshitz^[12] give for the critical mass a value which is 0.76 times the mass of the sun.

At present it is not known whether there is even one collapsed star in the universe. Nevertheless, in order to estimate the mean density of nucleons in the universe it is necessary to give an upper limit on the number of collapsed stars which have dimmed out and lost their mass, but have kept their original numbers of nucleons (we take the point of view of the unlimited validity of the law of conservation of the number of nucleons, or, more exactly, of the baryon charge). We can make such an estimate if we assume that these stars have arisen by evolution from ordinary hot

*Allowing for the fact that a partial conversion of nucleons into strange particles is possible at ultrahigh densities.^[10,11]

stars—accumulations of hydrogen. Then the energy that corresponds to the difference between the initial mass of the star (in a rarefied state) and its final mass (in the collapsed state) must have been removed from the star in some form or other.

Since the during release of the main part of the energy the density of the star is extremely large, the main mechanism is probably the emission of neutrinos and antineutrinos (cf. [7,13-15]). Even independently of the concrete form in which it is released, however, the energy is not lost, and can be detected by its gravitational action. Thus we can use the estimate of the preceding section and draw the conclusion that the mean density in the universe of nucleons in collapsed stars cannot exceed the known value of the density of ordinary nucleons by more than a factor of 10 to 20.

To our earlier stipulations about the assumptions on which our conclusions are based (isotropic model, and so on), we must add one more, namely, that the collapse did not occur too early; otherwise during the time from the collapse to the present the energy of the neutrinos could decrease through the adiabatic expansion of the neutrino gas in the expanding universe. If this were so, our estimate of an upper limit on the number of nucleons in collapsed stars would have to be raised.

¹B. M. Pontecorvo and Ya. A. Smorodinskii, JETP 41, 239 (1961), Soviet Phys. JETP 14, 173 (1962).

²Reines, Cowan, Harrison, McGuire, and Kruse, Phys. Rev. 117, 159 (1960).

³R. Davis, Jr. and D. S. Harmer, Bull. Amer. Phys. Soc. II, 4, 217 (1959).

⁴F. Reines, Ann. Rev. Nuclear Sci. 10, 1 (1960).

⁵B. M. Pontecorvo, JETP 37, 1751 (1959), Soviet Phys. JETP 10, 1236 (1960).

⁶A. Einstein, The Meaning of Relativity, Princeton Univ. Press, 1950, (Russ. Transl. p. 108).

⁷G. M. Gandel'man and V. S. Pinaev, JETP 37, 1072 (1959), Soviet Phys. JETP 10, 764 (1960).

⁸F. Hoyle, Proc. Phys. Soc. 77, 1 (1961).

⁹Ya. A. Smorodinskii, Trudy VII soveshchaniya po kosmogonii (Proc. 7th Conference on Cosmogony), Moscow, AN SSSR, 1957, page 131.

¹⁰Ya. B. Zel'dovich, JETP 37, 569 (1959), Soviet Phys. JETP 10, 403 (1960).

¹¹V. A. Ambartsumyan and G. S. Saakyan, Astrofiz. Zhurn. 37, 193 (1960).

¹²L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Gostekhizdat, 1951, page 353.

¹³B. M. Pontecorvo, JETP 36, 1615 (1959), Soviet Phys. JETP 9, 1148 (1959).

¹⁴H. Y. Chiu and P. Morrison, Phys. Rev. Letters 5, 573 (1960).

¹⁵M. Gell-Mann, Phys. Rev. Letters 6, 70 (1961).

CORRELATION OF ELECTRON AND POSITRON POLARIZATIONS IN RELATIVISTIC PAIRS

Ya. B. ZEL'DOVICH

Submitted to JETP editor April 14, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 912-913 (September, 1961)

An electron-positron pair for which the square of the sum of the four-momenta is much larger than m_e^2 consists either of a right-handed electron and a left-handed positron, or else of a left-handed electron and a right-handed positron; here the terms "left-handed" and "right-handed" refer to the longitudinal polarization of the particle. In the center-of-mass system of the pair the projection of the total spin of the pair on the direction of motion of one particle can have only two values, $+1$ and -1 , but not 0 . Processes in which this rule is violated have a probability which is smaller by a factor $(m_e/E)^2$ as compared with the probability of "allowed" processes.

FOR the proof of the statements made in the abstract, which apply to particles with energies many times the rest mass, we omit the term in the Dirac equation that contains the mass of the electron. It is well known that the Dirac equation then separates into two first-order equations for two two-component spinors. Physically this corresponds to the fact that for zero mass we can speak of right-handed and left-handed electrons as two independent particles which do not get converted into each other. Furthermore each type of particle has its antiparticles: the antiparticles of right-handed electrons are left-handed positrons, and those of left-handed electrons are right-handed positrons. In a recent paper by Sannikov^[1] these ideas have been used in a consistent way for the construction of the Green's function and the scattering matrix and for the calculation of scattering and radiation emission of high-energy electrons.

Still earlier, before the discovery of parity non-conservation and the two-component nature of the neutrino, the separation of the equation was used in the concrete problem of the scattering of fast electrons by a nucleus by Yennie, Ravenhall, and Wilson^[2]; they showed without calculation that the longitudinal polarization of the electron does not change sign in the scattering.

Let us now examine the qualitative consequences of the separation of the equation for the effect of pair production.

The production of a right-handed (left-handed) electron is necessarily accompanied by the production of a left-handed (right-handed) positron. For $m = 0$ the production of a right-handed electron and a right-handed positron, i.e., the produc-

tion of a particle with the wrong antiparticle, is just as impossible as the direct production of a μ^+e^- pair. Thus there is a correlation between the longitudinal polarizations of the two particles of a pair.

Let us consider a pair in the coordinate system in which $\mathbf{P} = \mathbf{p}_e^+ + \mathbf{p}_e^- = 0$. In this coordinate system the momenta of the electron and positron are antiparallel, and consequently the projections of the spins of the electron and positron on the direction of motion, say of the electron, must be either $+\frac{1}{2}$ and $+\frac{1}{2}$ (right-handed electron and left-handed positron), or else $-\frac{1}{2}$ and $-\frac{1}{2}$ (left-handed electron and right-handed positron). Since the projection of the orbital angular momentum on this direction is zero, the projection of the total angular momentum j of the pair is ± 1 .

Let us consider the process $\pi^+ + \pi^- \rightarrow \gamma \rightarrow e^+ + e^-$. The projection of the angular momentum j_π on the direction of motion of a π meson is zero. The projection of the angular momentum j_e of the pair on the direction of motion of e is ± 1 . Consequently, the direction of motion of the electron cannot coincide with the direction of motion of the meson. In fact, according to a formula derived by Akhiezer and Berestetskii,^[3]

$$d\sigma/d\Omega \sim 1 - (1 - m^2/E^2) \cos^2 \theta,$$

and for $m = 0$ we have $d\sigma/d\Omega = 0$ at $\theta = 0$.

In the approximation $m = 0$ the decay of a π^0 meson ($j = 0$) into a pair $e^+ + e^-$ is impossible. For m different from zero, Gell-Mann and Feynman^[4] propose formulating the theory as a second-order equation for a two-component quantity. For the following argument, however, it is more conven-

ient to deal with two first-order equations for two types of particles, with the mass playing the part of an interaction constant which causes the conversion of one type into the other.^[5] Then besides the order of a process in the constant $e^2/\hbar c$ we can speak of its order in the constant m . For example, the decay $\pi^0 \rightarrow e^+ + e^-$ is forbidden in zeroth order in m , but occurs in first order in m . Therefore it is indeed so^[6] that the probability of the process $\pi^0 \rightarrow e^+ + e^-$, taken as a fraction of the probability of the normal decay $\pi^0 \rightarrow 2\gamma$, is not of the order $(e^2/\hbar c)^2 = 10^{-4}$, but of the order $(e^2/\hbar c)^2 (m_e/m_\pi)^2 = 10^{-8}$. The situation in the decay $\pi^0 \rightarrow e^+ + e^-$ is the same as for the decays $\pi^+ \rightarrow e^+ + \nu$ and $\pi^- \rightarrow e^- + \bar{\nu}$, where because of conservation of angular momentum the charged particles are produced with the "incorrect" sign of the longitudinal polarization.

Pair production in 0-0 transitions of nuclei is allowed for $m = 0$, since owing to the recoil of the nucleus the angular momentum of the pair as a whole compensates the intrinsic angular momentum of the pair.

I take occasion to express my gratitude to A. V. Berkov, L. D. Landau, M. Ya. Ovchinnikova, L. B. Okun', and I. Ya. Pomeranchuk for discussions.

¹S. S. Sannikov, JETP **40**, 237 (1961), Soviet Phys. JETP **13**, 163 (1961).

²Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954).

³A. I. Akhiezer and V. B. Berestetskii, *Kvantovaya élektrodinamika* (Quantum Electrodynamics), 2nd edition, Fizmatgiz, 1959, page 645.

⁴R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

⁵Ya. B. Zel'dovich, JETP **40**, 637 (1961), Soviet Phys. JETP **13**, 444 (1961).

⁶S. M. Berman and D. A. Geffen, Nuovo cimento **18**, 1192 (1960).

Translated by W. H. Furry

ON $\Sigma^+ \rightarrow p + e^+ + e^-$ AND $\Sigma^+ \rightarrow p + \mu^+ + \mu^-$ DECAYS

I. V. LYAGIN and É. Kh. GINZBURG

Smolensk State Pedagogical Institute

Submitted to JETP editor April 14, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 914-918 (September, 1961)

An investigation of the process $\Sigma^+ \rightarrow p + e^+ + e^-$ is of interest in connection with the search for neutral currents in weak interactions. According to the theory of universal charged currents^[1] this decay cannot be induced by weak interaction alone, but it can proceed under the influence of strong and electromagnetic interaction. The total decay probability and recoil-proton energy spectrum are computed by taking into account this circumstance. An expression for the probability of the decay $\Sigma^+ \rightarrow p + \mu^+ + \mu^-$ is also obtained and the ratio of the two decay probabilities is estimated.

THREE cases of hyperon decay into a proton and a γ quantum

$$\Sigma^+ \rightarrow p + \gamma \quad (1)$$

were observed recently.

Decays of this type, and particularly the structure of the matrix element corresponding to the decay (1), were discussed by Behrends.^[2] These decays are due to a combination of strong, weak, and electromagnetic interactions, which can be represented, for example, by the Feynman diagram of Fig. 1, where the weak, strong, and electromagnetic interactions correspond to vertices a, b, and c respectively. However, since there is no theory of strong interactions at present, the process represented by this diagram cannot be calculated, decays such as (1) are examined phenomenologically, and diagrams such as this are replaced by a block A representing the totality of all the diagrams that contribute to the process under investigation (see Fig. 2). The general form of the matrix corresponding to the vertex $(\bar{p}\Sigma\gamma)$ in the matrix element will be

$$\Gamma^\alpha = (a_1 + a_2\gamma_5)\gamma^\alpha + (b_1 + b_2\gamma_5)(\gamma^\alpha\hat{q} - \hat{q}\gamma^\alpha) + (c_1 + c_2\gamma_5)q^\alpha, \quad (2)$$

where γ^α are Feynman matrices, q is the four-momentum of the photon,* and $a_1, \dots, c_2$ are certain functions of q^2 . In view of the gauge invariance we should have

$$g^{\alpha\alpha}q^\alpha\bar{u}_p\Gamma^\alpha u_\Sigma = 0,$$

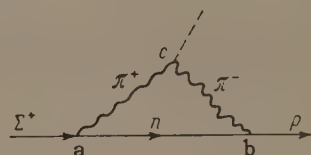


FIG. 1

thus bringing about a connection between a and c:

$$a_1 = c_1 q^2 / (m_\Sigma - m_p), \quad a_2 = c_2 q^2 / (m_\Sigma + m_p).$$

Since $q^2 = 0$ and $q\epsilon = 0$ for a real photon (ϵ is the photon polarization vector), the amplitude of the decay (1)

$$M_\gamma = \frac{G}{\sqrt{2}} g^{\alpha\alpha} \bar{u}_p \Gamma^\alpha u_\Sigma e^\alpha$$

depends only on b_1 and b_2 , and the decay probability is*

$$w_\gamma = 2G^2 (b_1^2 + b_2^2) \Delta^3 / \pi, \quad (3)$$

where G is the weak-interaction constant, $\Delta = (m_\Sigma^2 - m_p^2)/2m_\Sigma$ is the maximum photon energy, while m_Σ and m_p are respectively the masses of the Σ^+ hyperon and the proton. Thus, an investigation of this decay does not yield any information on the values of a and c .

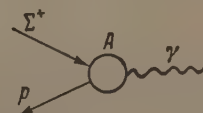


FIG. 2

*We use henceforth the matrices and symbols given in the book by Bogolyubov and Shirkov,^[3] and also a system of units in which $\hbar = c = 1$.

*As usual, we assume in first approximation that b_1 and b_2 are constants.

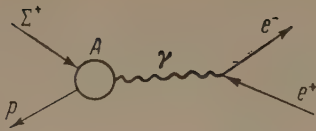


FIG. 3

To obtain effects in which a and c play a role, it is necessary to consider the hitherto unobserved decay with internal conversion of a γ quantum and production of an electron-positron pair

$$\Sigma^+ \rightarrow p + e^+ + e^-. \quad (4)$$

This decay corresponds to the diagram shown in Fig. 3 and to the matrix elements

$$M_{e^+e^-} = \frac{G}{\sqrt{2}} g^{\alpha\alpha} \bar{u}_p \Gamma^\alpha u_\Sigma q^{-2} \bar{u}_e \gamma^\alpha u_{e^+}, \quad (5)$$

and in this case the decay amplitude is already dependent on the values of a and c . An investigation of this process can thus yield some information on these constants. Standard calculations yield for the differential probability the following expression

$$dw = 2\pi \frac{G^2 e^2}{4} g^{\alpha\alpha} g^{\beta\beta} M^{\alpha\beta} \frac{1}{q^4} \{K_- K_+^\beta + K_-^\beta K_+^\alpha - g^{\alpha\beta} (K_- K_+ + m^2)\} \delta(-K_\Sigma + K_p + K_- + K_+) dK_p dK_- dK_+ / (2\pi)^6 K_-^0 K_+^0, \quad (6)$$

where m is the electron mass, K_Σ , K_p , K_- , and K_+ are the respective 4-momenta of the hyperon, proton, electron, and positron, and

$$M^{\alpha\beta} = \text{Sp} \{(\hat{K}_p + m_p) \Gamma^\alpha (K_\Sigma + m_\Sigma) \bar{\Gamma}^\beta\} / 4K_\Sigma^0 K_p^0$$

(we average over the variables of the Σ^+ hyperon and sum over the variables of the electron and positron).

In the integration with respect to $dK_- dK_+$ we change to new variables:^[4]

$$q = K_- + K_+, \quad r = K_- - K_+.$$

We thus obtain

$$\int \{K_- K_+^\beta + K_-^\beta K_+^\alpha - g^{\alpha\beta} (K_- K_+ + m^2)\} \frac{dK_- dK_+}{K_-^0 K_+^0} = \frac{1}{2} \int (q^\alpha q^\beta - r^\alpha r^\beta - g^{\alpha\beta} q^2) \delta(q^2 + r^2 - 4m^2) \delta(qr) dq dr.$$

The integral with respect to dr has the following tensor structure

$$\int (q^\alpha q^\beta - r^\alpha r^\beta - g^{\alpha\beta} q^2) \delta(q^2 + r^2 - 4m^2) \delta(qr) dr = A g^{\alpha\beta} + B q^\alpha q^\beta,$$

where A and B are certain functions of q^2 . To determine these functions we multiply both halves of this identity first by $g^{\alpha\beta}$ and then by $g^{\alpha\beta} q^\alpha q^\beta$, obtaining after a summing over the double index the following two equations

$$\int (-r^2 - 3q^2) \delta(q^2 + r^2 - 4m^2) \delta(qr) dr = 4A + Bq^2, \\ 0 = Aq^2 + Bq^4.$$

The integral in the left half of the first equation we calculate in a system in which $q = 0$, and upon changing to the invariant form we obtain

$$A = -Bq^2, \quad B = \frac{4\pi q^2 + 2m^2}{3q^2} \sqrt{\frac{q^2 - 4m^2}{q^2}}. \quad (7)$$

We thus have from (6) and (7)

$$dw = \frac{1}{3(2\pi)^5} G^2 e^2 (q^2 + 2m^2) \sqrt{\frac{q^2 - 4m^2}{q^2}} \frac{1}{q^4} \times g^{\alpha\alpha} g^{\beta\beta} M^{\alpha\beta} (-g^{\alpha\beta} + q^\alpha q^\beta q^{-2}) \delta(-K_\Sigma + K_p + q) dq dK_p. \quad (8)$$

After summing over the double index and eliminating the δ function, using the conservation law $m_\Sigma - m_p = q$, we make the following substitution in (8):

$$\frac{dK_p}{K_p^0} = -4\pi \frac{q dq}{m_\Sigma} |K_p| = -4\pi \left[\left(\frac{\Delta^2 - q^2}{2m_\Sigma} \right)^2 + \frac{m_p}{m_\Sigma} (\Delta^2 - q^2) \right]^{1/2} \frac{q dq}{m_\Sigma}, \quad (9)$$

where $\Delta = m_\Sigma - m_p$, and the factor 4π is obtained by integration over the angles.

To obtain an expression for the total probability we integrate (8) with respect to dq from $q_{\min} = 2m$ to $q_{\max} = \Delta$. In the limiting case of small values of Δ/m_Σ (in our case $\sim 1/5$) we obtain, accurate to terms proportional to Δ^2/m_Σ^2 inclusive,

$$w = \frac{1}{3} G^2 e^2 \frac{\Delta^3}{\pi^3} \left\{ (b_1^2 + b_2^2) \left[\ln \frac{2\Delta}{m} - \frac{23}{12} - \frac{3}{2} \frac{\Delta}{m_\Sigma} \left(\ln \frac{2\Delta}{m} - \frac{23}{12} \right) \right] + \frac{7}{8} \frac{\Delta^2}{m_\Sigma^2} \left(\ln \frac{2\Delta}{m} - \frac{847}{420} \right) - \frac{1}{4} (b_1^2 - b_2^2) \left[1 - \frac{3}{2} \frac{\Delta}{m_\Sigma} \right] + \frac{23}{40} \frac{\Delta^2}{m_\Sigma^2} \right\} + \frac{1}{8} \left(c_1^2 + \frac{\Delta^2 c_2^2}{4m_\Sigma^2 (1 - \Delta/2m_\Sigma)^2} \right) \left[\ln \frac{2\Delta}{m} - 2 - \frac{3}{2} \frac{\Delta}{m_\Sigma} \left(\ln \frac{2\Delta}{m} - 2 \right) + \frac{7}{8} \frac{\Delta^2}{m_\Sigma^2} \left(\ln \frac{2\Delta}{m} - \frac{221}{105} \right) - \frac{1}{40} \left(c_1^2 - \frac{\Delta^2 c_2^2}{4m_\Sigma^2 (1 - \Delta/2m_\Sigma)^2} \right) \left[1 - \frac{1}{2} \frac{\Delta}{m_\Sigma} - \frac{1}{56} \frac{\Delta^2}{m_\Sigma^2} \right] + \frac{1}{4} \left(b_1 c_1 - \frac{\Delta b_2 c_2}{2m_\Sigma (1 - \Delta/2m_\Sigma)} \right) \left[1 - \frac{6}{5} \frac{\Delta}{m_\Sigma} - \frac{1}{8} \frac{\Delta^2}{m_\Sigma^2} \right] - \frac{1}{4} \left(b_1 c_1 + \frac{\Delta b_2 c_2}{2m_\Sigma (1 - \Delta/2m_\Sigma)} \right) \left[1 - \frac{9}{5} \frac{\Delta}{m_\Sigma} - \frac{3}{8} \frac{\Delta^2}{m_\Sigma^2} \right] \right\}. \quad (10)$$

From (3) and (10) we can find the internal-conversion coefficient

$$\rho = w(\Sigma^+ \rightarrow p + e^+ + e^-) / w(\Sigma^+ \rightarrow p + \gamma).$$

To estimate the order of magnitude of ρ we limit ourselves in (10) to the first term, and get

$$\rho = \frac{2\alpha}{3\pi} \left(\ln \frac{2\Delta}{m} - \frac{23}{12} \right) \approx \frac{1}{130}. \quad (11)$$

Deviations from this value can be due to corrections for the structure of the matrix element and for the dependence of the form factors on q^2 . Thus, for example, if we take into account the second term in (10), which is proportional to $b_1^2 - b_2^2$, we obtain for the variation of ρ the following estimate

$$\frac{2\alpha}{3\pi} \ln \frac{2\Delta}{m} - \delta < \rho < \frac{2\alpha}{3\pi} \ln \frac{2\Delta}{m} + \delta, \quad \delta = \frac{1}{4} \frac{2\alpha}{3\pi}.$$

To estimate the influence of the dependence of the form factors on q^2 we put, for example,

$$b^2(q^2) = b_0^2 \left(1 - \frac{1}{6} q^2 r^2\right),$$

where $r \approx m_\pi/2 \approx 1/\Delta$ is the interaction radius (m_π is the pion mass). We then obtain

$$w = \frac{1}{3} G^2 e^2 \frac{\Delta^3}{\pi^3} (b_{01}^2 + b_{02}^2) \left(\ln \frac{2\Delta}{m} - \frac{179}{90} \right).$$

From this we conclude that this influence is small.

We note that expression (11) is in the nature of the transverse part of the conversion coefficient, since the longitudinal part due to the term $q^\alpha q^\beta / q^2$ in (8) is independent of $(b_1^2 + b_2^2)$ and $(b_1^2 - b_2^2)$ (see [5]).

Let us consider also the recoil-proton momentum distribution obtained from (6) after integrating over the electron and positron momenta. Neglecting in this expression terms that result in a small correction, we obtain

$$dw = C' \sqrt{\Delta^2 - q^2} \sqrt{q^2 - 4m^2} q^{-2} dq.$$

The factor C' depends here on the form factors. From the conservation law we obtain (accurate to Δ/m_Σ)

$$dw = C'' K_p^2 \sqrt{\Delta^2 - K_p^2} (\Delta^2 - K_p^2)^{-3/2} d|K_p|.$$

Let us put $x^2 = K_p^2 / \Delta^2$ and $a = 2m/\Delta$. Then

$$dw = C f(x) dx,$$

$$f(x) = x^2 \sqrt{1 - a^2 - x^2} (1 - x^2)^{-3/2}, \quad 0 \leq x^2 \leq 1 - a^2.$$

$f(x)$ has a sharp maximum at $x_{\max} = (2 - 2a^2) / (2 + a^2)$, and $f(x_{\max}) \approx 2 (\Delta/2m)^2$ (see Fig. 4). This means that the decays occur for the most part with formation of "narrow" pairs, wherein the proton acquires a maximum momentum.

From (8) and (9) we can obtain, under the same assumptions with respect to the structure of the matrix element, an expression for the total decay probability

$$\Sigma^+ \rightarrow p + \mu^+ + \mu^-. \quad (12)$$

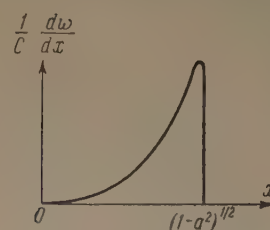


FIG. 4

Recognizing that the muon mass is large (compared with the electron mass), we can modify slightly the method of integrating with respect to dK_p in (8) and confine ourselves to terms proportional to $(b_1^2 + b_2^2)$ and $(b_1^2 - b_2^2)$. We then obtain, with accuracy to $m_\mu^2/\Delta^2 \approx 0.17$ inclusive,

$$w = \frac{1}{3} G^2 e^2 \frac{\Delta^3}{\pi^3} \cdot \frac{\pi}{2} \left\{ (b_1^2 + b_2^2) \left(\frac{13}{8} \frac{m_\mu^2}{\Delta^2} - \frac{1}{16} \right) - \frac{3}{4} (b_1^2 - b_2^2) \left(\frac{1}{4} - \frac{m_\mu^2}{\Delta^2} \right) \right\},$$

where m_μ is the muon mass. For the ratio of the probabilities of decays (4) and (12) we obtain the estimate

$$\frac{w(\Sigma^+ \rightarrow p + e^+ + e^-)}{w(\Sigma^+ \rightarrow p + \mu^+ + \mu^-)} \approx \frac{\ln(2\Delta/m) - 23/12}{\pi (13m_\mu^2/8\Delta^2 - 1/16)/2} \approx 14.5.$$

The authors are deeply grateful to L. B. Okun' for suggesting the topic and for discussions.

¹R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

²R. E. Behrends, Phys. Rev. **111**, 1691 (1958).

³N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience, 1959.

⁴R. H. Dalitz, Phys. Rev. **99**, 915 (1955).

⁵N. M. Kroll and W. Wade, Phys. Rev. **98**, 1355 (1955).

Translated by J. G. Adashko

ANALYSIS OF THE DISTRIBUTIONS OF THE TRANSVERSE MOMENTA OF PIONS AND STRANGE PARTICLES

N. N. ROINISHVILI

Institute of Physics, Academy of Sciences, Georgian S.S.R.

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 919-923 (September, 1961)

Submitted to JETP editor April 15, 1961

The experimental distributions of the transverse momenta of π mesons and strange particles are analyzed on the basis of various theoretical models. According to the quasi-unidimensional hydrodynamic theory, π mesons and strange particles should be emitted at the same temperature $kT \approx m_\pi c^2$, whereas according to the field theory the radius of the region of π -meson production should be approximately twice the size of the region of hyperon and K-meson emission.

MANY recently published papers are devoted to an interpretation of the distribution of the transverse momenta of shower particles from the point of view of various theoretical representations.^[1-3] From an analysis of these papers it follows that apparently the distribution of $p_\perp$ is not very sensitive to the type of interaction responsible for multiple generation. Nonetheless, information on the $p_\perp$ spectra can prove to be quite useful, for through the use of certain theoretical concepts one can obtain, with good accuracy, the values of certain physical parameters of interest to the theory under consideration from the distribution of the transverse momenta. We shall attempt to analyze the spectra of the transverse momenta of pions and strange particles from the point of view of two interaction models, which in a certain sense are contrary.

1. Let us assume that the process of multiple generation of particles is described by a quasi-unidimensional variant^[1] of the hydrodynamic theory. In this case, owing to the absence of hydrodynamic liquid flow in the transverse direction, the $p_\perp$ distribution will have the form

$$\frac{dN}{N} \frac{dp_\perp}{p_\perp} = \frac{p_\perp}{(mc)^2} \frac{mc^2}{kT} \sqrt{\left(\frac{p_\perp}{mc}\right)^2 + 1} \times \frac{\sum_{n=0}^{\infty} (\mp 1)^{n-1} K_1 \left(n \frac{mc^2}{kT} \sqrt{\left(\frac{p_\perp}{mc}\right)^2 + 1} \right)}{\sum_{n=0}^{\infty} (\mp 1)^{n-1} K_2 (nmc^2/kT)/n} \quad (1)$$

and will be characterized by a mean-square value

$$\langle p_\perp^2 \rangle = 2 \frac{kT}{mc^2} \frac{\sum_{n=0}^{\infty} (\mp 1)^{n-1} K_2 (nmc^2/kT)/n^2}{\sum_{n=1}^{\infty} (\mp 1)^{n-1} K_2 (nmc^2/kT)/n} \quad (2)$$

where T is the dispersal temperature of the system, m is the mass of the investigated particle, $K_1(x)$, $K_2(x)$, and $K_3(x)$ are Bessel functions of imaginary argument, and the $\pm$ signs correspond to Bose and Fermi particles.

Comparing the spectrum^[1] at different values of the dispersal temperature with the experimental distribution of $p_\perp$ for pions^[2,4,5] and for a mixture of Λ^0 , θ^0 , and $\Sigma^\pm$ particles^[6] (Figs. 1a and b) we readily see that the hydrodynamic theory describes satisfactorily both spectra at a temperature kT on the order of the pion rest mass.

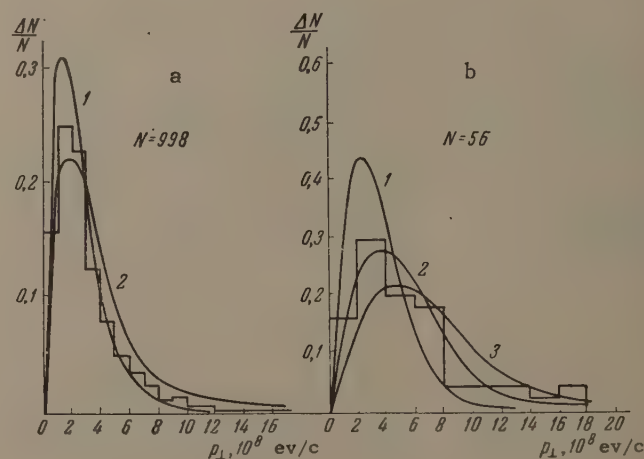


FIG. 1. a — $p_\perp$ distribution of pions registered in penetrating cosmic-ray showers with the aid of magnetic cloud chambers.^[2,4,5] Curves 1 and 2 correspond to $kT/m_\pi c^2 = 1$ and 1.5. b — $p_\perp$ distribution of mixture of strange particles registered in penetrating cosmic-ray showers with the aid of a magnetic cloud chamber.^[6] Curves 1, 2, and 3 correspond to $kT/m_\pi c^2 = 0.5, 1$, and 1.5. N — number of particles included in the histogram.

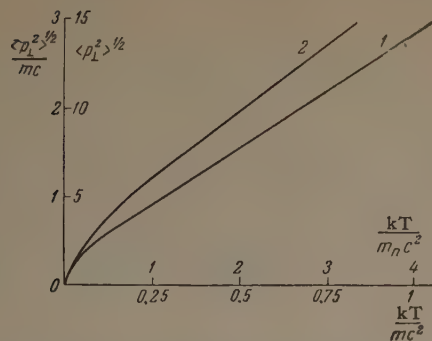


FIG. 2. Curve 1 — dependence of $\langle p_{\perp}^2 \rangle^{1/2} / mc$ on kT / mc^2 . Curve 2 — dependence of $\langle p_{\perp}^2 \rangle^{1/2}$ on $kT / m_{\pi} c^2$ for the mixture of Λ^0 , θ^0 , and $\Sigma^{\pm}$ observed in [6]. The transverse momenta are in units of 10^8 ev/c.

We can determine the dispersal temperature numerically by using the mean-square momenta $p_{\perp}$ of pions and strange particles with the aid of the curves shown in Fig. 2. Curve 1 corresponds to the dependence of $\langle p_{\perp}^2 \rangle^{1/2}$ on kT for bosons of arbitrary mass, since both variables are plotted on this curve in mass units. At temperatures kT several times smaller than the nucleon mass, the curve can also be applied to baryons. Curve 2 represents the same dependence of $\langle p_{\perp}^2 \rangle^{1/2}$ on kT for the mixture of strange particles observed in [6] (46% Λ^0 , 32% θ^0 , and 23% $\Sigma^{\pm}$). In Fig. 2 the value of $\langle p_{\perp}^2 \rangle^{1/2}$ is in units of 10^8 ev/c, and kT is given in pion masses.

The mean-squared momentum $p_{\perp}$ corresponding to the histogram 1a has a value

$$\langle p_{\perp}^2 \rangle^{1/2} = (3.86^{+0.48}_{-0.05}) \cdot 10^8 \text{ ev/c},$$

Assuming that all the particles included in the distribution are pions, we obtain from curve 1 of Fig. 2 a critical temperature

$$kT = (0.96^{+0.14}_{-0.01}) m_{\pi} c^2.$$

An analogous estimate can be made also for the dispersal temperature of strange particles, using curve 2 of Fig. 2 and the experimental value of $\langle p_{\perp}^2 \rangle^{1/2}$ corresponding to the histogram 1b:

$$\langle p_{\perp}^2 \rangle^{1/2} = (6.56^{+2.01}_{-0.39}) 10^8 \text{ ev/c},$$

which leads to

$$kT = (1.1^{+0.6}_{-0.1}) m_{\pi} c^2.$$

We see thus that the $p_{\perp}$ distribution gives a sufficiently accurate value of the dispersal temperature of pions and strange particles, provided that the hydrodynamic theory fits the process of multiple generation of particles at high energies.

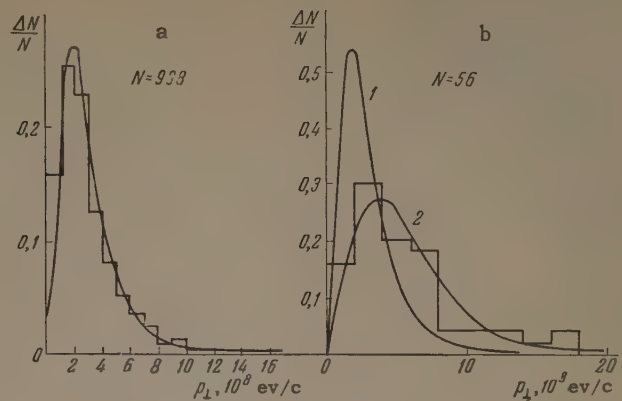


FIG. 3. a — $p_{\perp}$ distribution of pions. The curve corresponds to an exponential distribution of source density with $\langle r^2 \rangle^{1/2} = 0.8 \times 10^{-13}$ cm. b — $p_{\perp}$ distribution of a mixture of strange particles. Curves 1 and 2 correspond to an exponential distribution of the source density with $\langle r^2 \rangle^{1/2} = 0.8 \times 10^{-13}$ cm and $\langle r^2 \rangle^{1/2} = 0.4 \times 10^{-13}$ cm respectively.

2. In the Heisenberg theory, [7] however, $\langle p_{\perp}^2 \rangle^{1/2}$ determines the transverse dimension of the region where particles are generated, in accordance with the uncertainty principle

$$\langle p_{\perp}^2 \rangle^{1/2} \langle r_{\perp}^2 \rangle^{1/2} \geq \hbar.$$

The values which we have used for the mean squares of the pion and strange-particle momenta lead to a certain structural representation of the region in which the radius of the volume from which pions are emitted is

$$\langle r_{\perp}^2 \rangle^{1/2} \geq 0.52 \cdot 10^{-13} \text{ cm},$$

and the radius for strange particles is

$$\langle r_{\perp}^2 \rangle^{1/2} \geq 0.30 \cdot 10^{-13} \text{ cm}.$$

Neglecting the interaction of the emitted particles (which makes the following analysis essentially contradict the hydrodynamic analysis), we can relate not only $\langle p_{\perp}^2 \rangle^{1/2}$ but the entire distribution of the transverse momenta with the structural generation region* by assuming a) that there exists a certain coordinate system in which the momentum particle spectrum $|\varphi(p)|^2$ is independent of the angle of emission and b) that the connection between the momentum spectrum in this coordinate system and the probability density $|\rho(r)|^2$ of observing a particle at a distance r from the center of the region is given by the Fourier integral

$$\varphi(p) = \int \rho(r) e^{-ipr} dr. \quad (3)$$

*A similar attempt is cited by Nagai and Ito. [8]

By specifying the form of the function $|\rho(r)|^2$ and integrating the resultant function $\varphi(p)$ with respect to the longitudinal momentum $p_{||}$ we can obtain the distribution over $p_{\perp}$ and compare it with the experimental histogram. It turns out that for many functions by suitable choice of the parameters (in particular, $\langle r_{\perp}^2 \rangle^{1/2}$) we can reconcile the resultant $p_{\perp}$ spectra with those experimentally observed. Agreeing best with the distribution of the transverse pion momenta is a probability density that decreases exponentially with the distance

$$|\rho(r)|^2 = \exp [-(12/\langle r^2 \rangle)^{1/2} r] \quad (4)$$

with a parameter

$$\langle r^2 \rangle^{1/2} = 0.8 \cdot 10^{-13} \text{ cm.}$$

It is interesting to note that the function (4) with the same value of $\langle r^2 \rangle^{1/2}$ describes best of all the electromagnetic structure of protons investigated by scattering electrons with energies up to 550 Mev.^[9]

If it is assumed that the function $|\rho(r)|^2$ has the same form for strange particles, the corresponding mean-squared radius has approximately half the value, about 0.4×10^{-13} cm.

The foregoing is illustrated in Figs. 3a and b, on which the $p_{\perp}$ distribution corresponding to an exponential density is compared with the histogram of the transverse momenta of pions and strange particles.

3. The foregoing analysis of the $p_{\perp}$ distributions leads to the following conclusions:

a) No preference can be given to any of the multiple-generation theories on the basis of the distribution of the transverse momenta of pions and strange particles.

b) From the point of view of the quasi-unidimensional hydrodynamic constant, the $p_{\perp}$ spectra of pions and strange particles show that the system breaks up at a temperature $kT \approx m_{\pi}c^2$.

c) If the field-theoretical description of the process of multiple generation is correct, it must

be assumed that the region for the emission of strange particles has approximately half the radius as that for the generation of pions.

d) Finally, if it is assumed that the amplitudes of the momentum spectrum of the generated particles and of the spatial distribution of the source can be related by a Fourier integral, then the $p_{\perp}$ spectrum of the pions is in best agreement with an exponential distribution of the source density, with $\langle r^2 \rangle^{1/2} = 0.8 \times 10^{-13}$ cm.

The author takes this pleasant opportunity to thank Prof. É. L. Andronikashvili and E. L. Feinberg, and also S. G. Matinyan, É. V. Gedalin, and V. N. Roïnishvili for valuable advice in the discussion of the results.

¹G. A. Milekhin and I. L. Rozental', JETP **33**, 197 (1957), Soviet Phys. JETP **6**, 154 (1958).

²L. F. Hansen and W. B. Fretter, Phys. Rev. **118**, 812 (1960).

³E. G. Boos and Zh. S. Takibaev, JETP **38**, 1276 (1960), Soviet Phys. JETP **11**, 920 (1960).

⁴Birger, Grigorov, Guseva, Zhdanov, Slavatskii, and Stashkov, JETP **31**, 971 (1956), Soviet Phys. JETP **4**, 872 (1957).

⁵Awunor-Renner, Blaskovith, French, Chesquiere, Minvielle-Devaux, Neale, Pelletier, Rivet, Sahiar, and Skillicon, Nuovo cimento **17**, 134 (1960).

⁶É. L. Andronikashvili and N. N. Roïnishvili, JETP **39**, 265 (1960), Soviet Phys. JETP **12**, 190 (1961).

⁷W. Heisenberg, Z. Physik **133**, 65 (1952); Kosmische Strahlung, Berlin-Heidelberg-Göttingen, (1953).

⁸H. Nagai and D. Ito, Prog. Theoret. Phys. **23**, 966 (1960).

⁹R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

Translated by J. G. Adashko

ANALYTIC PROPERTIES OF A SQUARE DIAGRAM WITH NON-DECAYING MASSES

V. N. GRIBOV, G. S. DANILOV, and I. T. DYATLOV

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 22, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 924-936 (September, 1961)

An explicit form of the dispersion representations in energy and momentum transfer for the simplest square diagram with non-decaying masses is derived.

THE analytic properties of the simplest square diagram (four-branch vertex) have been investigated by many authors.^[1] The aim of the present paper is to determine the explicit form of the dispersion representations in the energy s and the square of the momentum transfer t in the center-of-mass system for the simplest square diagram with arbitrary external mass values consistent with the stability condition.

If the external masses are sufficiently small, one can write the amplitude under consideration as a dispersion integral of the imaginary part, which can be determined with the help of the unitarity condition. As the masses of the external particles are increased, anomalous terms appear in the dispersion representation; these are due to the anomalous thresholds or to the presence of complex singularities in the amplitude. These terms can be determined by analytic continuation with respect to the masses appearing in the dispersion relations. The method of analytic continuation with respect to the masses, as proposed by Mandelstam,^[2] has been used earlier in the investigation of the anomalies of the square diagram with different masses.^[3]

In the first section of this paper we shall consider the singularities of the imaginary part of the amplitude, which must be known for the analytic continuation with respect to the masses in the dispersion relations. In Sec. 2 we shall carry out the analytic continuation of the dispersion integral over s with respect to one of the masses with the condition $t < 0$. In Sec. 3 we shall obtain dispersion representations which are valid for arbitrary values of t by analytic continuation with respect to t . We shall see that the amplitude has a complex singularity in s for certain nonphysical values of t . It is interesting to note that this singularity lies in the lower half-plane of s . Thus the amplitude is an analytic function of s in the upper half-plane for arbitrary mass values and for arbitrary real

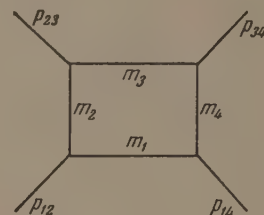


FIG. 1

t . This result remains valid when the external masses do not satisfy the stability condition, as will be shown in a subsequent paper.

In Sec. 4 we carry out the analytic continuation with respect to the masses of the other particles. In this way we obtain a dispersion representation in s for arbitrary external mass values satisfying the stability condition. It follows from the formulas of the present paper that the anomalous term can in all cases be written as an integral of a function which is formally identical with the Mandelstam function $A_{12}(s, t)$.^[4] The dispersion representation in t can obviously be obtained from the representation in s by making the replacement $s \leftrightarrow t$ and relabeling the masses.

1. FORMULATION OF THE PROBLEM. POSITION OF THE SINGULARITIES OF THE IMAGINARY PART OF THE AMPLITUDE

Let us consider the anomalous singularities of the diagram shown in Fig. 1. Let $p_{12} = p_{21}$, $p_{23} = p_{32}$, $p_{34} = p_{43}$, $p_{14} = p_{41}$ be the four-momenta of the particles involved in the reaction, and let m_1^2 , m_2^2 , m_3^2 , and m_4^2 be the squares of the masses of the virtual particles. The amplitude corresponding to the diagram of Fig. 1 can be written in the form

$$A = \frac{ig^4}{32(2\pi)^{18}(2p_{12}^0 \cdot 2p_{23}^0 \cdot 2p_{34}^0 \cdot 2p_{14}^0)^{1/2} m_1 m_2 m_3 m_4} A(\mu_{ik}), \quad (1)$$

where g is the coupling constant and the μ_{ik} are given by*

*The notation is the same as in [5].

$$\mu_{ik} = \mu_{ki} = (m_i^2 + m_k^2 - p_{ik}^2)/2m_i m_k,$$

with

$$p_{13}^2 = p_{31}^2 = (p_{12} + p_{23})^2 = s, \quad p_{24}^2 = p_{12}^2 = (p_{12} + p_{14})^2 = t.$$

It follows from the definition of μ_{ik} that in the physical region of the reaction $(p_{12}, p_{23}) \rightarrow (p_{34}, p_{14})$

$$\mu_{13} < -1, \quad \mu_{24} > 1.$$

The function $A(\mu_{ik})$ is an analytic function of each of its arguments if the other arguments are fixed. It is known that for sufficiently small internal masses, the function $A(\mu_{ik})$ (with $\mu_{24} > 1$) can be written as a dispersion integral over μ_{13} :

$$A(\mu_{ik}) = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{A_1}{\mu_{13} - \mu_{13}'} d\mu_{13}', \quad (2)$$

where $A_1 = \text{Im } A(\mu_{ik})$ is determined from the unitarity condition. After rather involved calculations one finds that

$$A_1 = K'^{-1/2} \ln(\xi_1/\xi_2), \quad \xi_1 = V - \sqrt{K'}, \quad \xi_2 = V + \sqrt{K'}, \quad (3)$$

where $K' = (\mu_{13}^2 - 1)K$, and V and K have the form

$$V = \mu_{24}(\mu_{13}^2 - 1) - \mu_{13}(\mu_{12}\mu_{34} + \mu_{23}\mu_{14}) + \mu_{23}\mu_{34} + \mu_{12}\mu_{14}, \quad (4)$$

$$K = \begin{vmatrix} 1 & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{12} & 1 & \mu_{23} & \mu_{24} \\ \mu_{13} & \mu_{23} & 1 & \mu_{34} \\ \mu_{14} & \mu_{24} & \mu_{34} & 1 \end{vmatrix}. \quad (5)$$

We assume in formula (3) that $\arg \xi_1 = \arg \xi_2 = 0$ and $\sqrt{K} > 0$ in the physical region.

The function $A(\mu_{ik})$ in formula (1) is the limit of the analytic function (2) obtained by letting μ_{13} approach the real axis from the lower half-plane.

In order to obtain for $A(\mu_{ik})$ a dispersion relation which is valid for large external masses, we must continue formula (2) analytically with respect to the masses. For this purpose we must investigate the positions of the singularities of A_1 in the variable μ_{13} as functions of the external masses. The singular points of A_1 are evidently determined by the equations

$$\xi_1 = 0, \quad (6)$$

$$\xi_2 = 0, \quad (7)$$

$$K = 0. \quad (8)$$

It is easily verified that (6) and (7) imply

$$\lambda(\mu_{12}, \mu_{23}, \mu_{13}) = 0, \quad (9)$$

$$\lambda(\mu_{34}, \mu_{14}, \mu_{13}) = 0, \quad (10)$$

where

$$\lambda(\alpha, \beta, \gamma) = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma - 1. \quad (11)$$

Equations (8) – (10) are the same as those obtained by applying the Landau method^[6] for the determination of the singular points to the diagram under consideration.

The solution of (9) and (10) can be written in the form

$$\Delta^\pm(\mu_{12}, \mu_{23}) = \mu_{12}\mu_{23} \pm \sqrt{(1 - \mu_{12}^2)(1 - \mu_{23}^2)}, \quad (12)$$

$$\Delta^\pm(\mu_{34}, \mu_{14}) = \mu_{14}\mu_{34} \pm \sqrt{(1 - \mu_{14}^2)(1 - \mu_{34}^2)}. \quad (13)$$

If the absolute values of μ_{12} , μ_{23} , μ_{14} , and μ_{34} are smaller than unity, then

$$\Delta^\pm(\mu_{12}, \mu_{23}) = \cos(\theta_{12} \pm \theta_{23}), \quad (14)$$

$$\Delta^\pm(\mu_{14}, \mu_{34}) = \cos(\theta_{14} \pm \theta_{34}), \quad (15)$$

where

$$\theta_{ik} = \arccos \mu_{ik}, \quad 0 \leq \theta_{ik} \leq \pi. \quad (16)$$

We shall denote the four roots of (9) and (10) also by the symbols $\Delta_i(\mu_{13})$ ($i = 1, 2, 3, 4$), numbering them in the order of increasing values:

$$\Delta_1(\mu_{13}) < \Delta_2(\mu_{13}) < \Delta_3(\mu_{13}) < \Delta_4(\mu_{13}).$$

The solution of (8) has the form

$$\mu_{13}^\pm = (\mu_{24}^2 - 1)^{-1} \{ \omega(\mu_{24}, \mu_{14}, \mu_{23}, \mu_{12}, \mu_{34}) \pm [\lambda(\mu_{12}, \mu_{14}, \mu_{24}) \lambda(\mu_{23}, \mu_{34}, \mu_{24})]^{1/2} \}. \quad (17)$$

Solving (8) with respect to μ_{24} , we have

$$\mu_{24}^\pm = (\mu_{13}^2 - 1)^{-1} \{ \omega(\mu_{13}, \mu_{23}, \mu_{14}, \mu_{12}, \mu_{34}) \pm [\lambda(\mu_{12}, \mu_{23}, \mu_{13}) \lambda(\mu_{34}, \mu_{14}, \mu_{13})]^{1/2} \}. \quad (18)$$

Here

$$\omega(\alpha, \beta, \gamma, \delta, v) = \alpha(\beta\gamma + \delta v) - (\beta v + \gamma\delta). \quad (19)$$

The functions $\mu_{13}^\pm$ and $\mu_{24}^\pm$ will also be denoted by $\square_i(\mu_{24})$ and $\square_i(\mu_{13})$, where $i = 1, 2$ and

$$\square_1(\mu_{24}) < \square_2(\mu_{24}), \quad \square_1(\mu_{13}) < \square_2(\mu_{13}).$$

It is seen from (17) and (18) that the curves (8) lie in the dashed regions of the (μ_{13}, μ_{24}) plane (see Fig. 2). For brevity, curves lying in the regions I, ..., V will in the following be called curves I, ..., V, respectively.

The curves I – IV have as asymptotes the straight lines $\mu_{13} = \pm 1$, $\mu_{24} = \pm 1$. These curves either lie entirely inside the region between the asymptotes or touch the straight lines $\Delta_1(\mu_{13})$, $\Delta_4(\mu_{13})$, $\Delta_1(\mu_{24})$, $\Delta_4(\mu_{24})$, where $\Delta_i(\mu_{24})$ are the roots of the equations

$$\lambda(\mu_{12}, \mu_{14}, \mu_{24}) = 0, \quad \lambda(\mu_{23}, \mu_{34}, \mu_{24}) = 0, \quad (20)$$

i.e.,

$$\mu_{24} = \Delta^\pm(\mu_{12}, \mu_{14}), \quad \mu_{24} = \Delta^\pm(\mu_{14}, \mu_{34}),$$

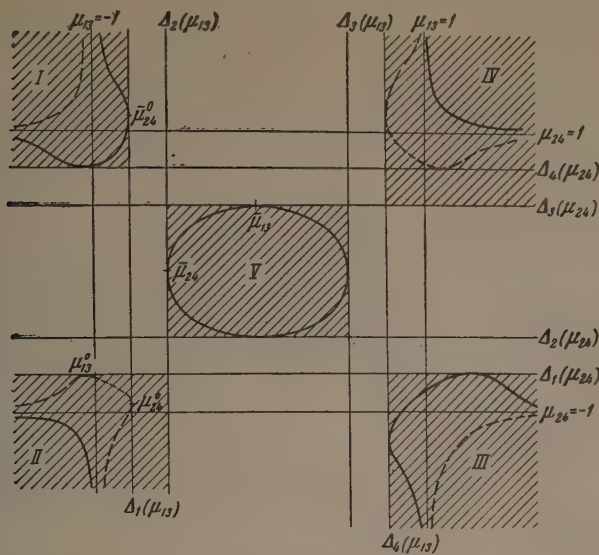


FIG. 2

numbered in the order of increasing values.

It can be shown that curve I touches the straight line $\Delta_1(\mu_{13})$ if the condition

$$(\mu_{12} + \mu_{23})(\mu_{14} + \mu_{34}) > 0 \quad (21)$$

is fulfilled. If the inequality sign is reversed, curve II touches the straight line $\Delta_1(\mu_{13})$.

If

$$(\mu_{12} + \mu_{14})(\mu_{23} + \mu_{34}) > 0, \quad (22)$$

curve III touches the straight line $\Delta_1(\mu_{24})$, whereas curve II touches the straight line $\Delta_1(\mu_{24})$ if the inequality sign in (22) is reversed. The position of the curves I—III under the conditions (21) and (22) is shown in Fig. 2 (solid lines). Figure 2 also shows the position of these curves when the conditions (21) and (22) are reversed (dotted lines).

It is easily seen that part of curve I can lie in the physical region for certain values of the masses. But this part of the curve is actually not singular for A_1 .

As will become clear in the following, formula (2) is valid if

$$\mu_{12} + \mu_{23} > 0, \quad \mu_{14} + \mu_{34} > 0, \quad (23)$$

i.e., if

$$\theta_{12} + \theta_{23} < \pi, \quad \theta_{14} + \theta_{34} < \pi.$$

Condition (23) implies the inequality (21). The positions of the curves I and II for this case are indicated by solid lines in Fig. 2.

2. ANALYTIC CONTINUATION WITH RESPECT TO μ_{12} AND μ_{23} FOR $\mu_{24} > 1$

Let us now turn to the analytic continuation of formula (2) with respect to μ_{12} in the region μ_{24}

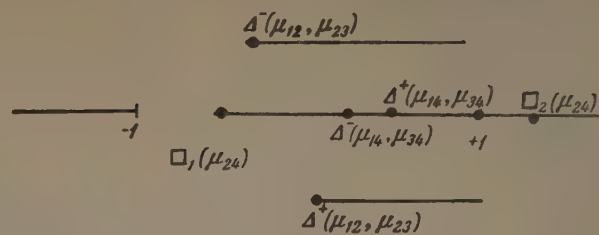


FIG. 3

> 1 . A dispersion relation for $A(\mu_{ik})$ for other values of μ_{24} can be obtained by analytic continuation with respect to μ_{24} .

In order to carry out the analytic continuation with respect to μ_{12} , we assume that μ_{12} has an infinitesimal imaginary part $-i\delta$. To be definite, we assume that $\delta > 0$. The case $\delta < 0$ is treated analogously. It can be shown that, for $\delta > 0$, the point $\Delta^-(\mu_{12}, \mu_{23})$ moves into the upper half-plane of μ_{13} , whereas $\Delta^+(\mu_{12}, \mu_{23})$ moves into the lower half-plane. If the mass p_{12}^2 is sufficiently small, the point $\square_1(\mu_{24})$ lies in the lower half-plane, and the point $\square_2(\mu_{24})$, in the upper half-plane. The positions of the singular points in the complex μ_{13} plane and the cuts leading away from them are shown in Fig. 3 for the case where the masses are sufficiently small, i.e., condition (23) is fulfilled.

The cuts coming from the singular points must be chosen in such a way that $\sqrt{K}$ is positive in the region $\mu_{13} < (-1, \square_1(\mu_{24}))$ and $\arg \xi_1 = \arg \xi_2 = 0$.

As p_{12}^2 increases, i.e., as μ_{12} decreases, the point $\Delta^-(\mu_{12}, \mu_{23})$ remains in the upper half-plane until μ_{12} reaches the value

$$\mu_{12} = -\mu_{23}, \quad (24)$$

i.e., $\theta_{12} + \theta_{23} = \pi$.

For small $\mu_{12} < -\mu_{23}$ the sign of the inequality (21) is reversed. The positions of the curves I and II for this case are indicated in Fig. 2 by dotted lines.

If $\mu_{12} = -\mu_{23}$, $\Delta^-(\mu_{12}, \mu_{23})$ hits the half-axis $\mu_{13} < -1$ in the point

$$\Delta^-(-\mu_{23}, \mu_{23}) = -1 - a\delta^2,$$

where a is some positive number. As μ_{12} decreases further, the point $\Delta^-(\mu_{12}, \mu_{23})$ moves into the lower half-plane. The point $\square_1(\mu_{24})$ moves into the upper half-plane for a certain $\mu_{12} > -\mu_{23}$, without touching the half-axis $\mu_{13} < -1$. The points $\Delta^+(\mu_{12}, \mu_{23})$ and $\square_2(\mu_{24})$ are of no interest, since they are always far removed from the half-axis $\mu_{13} < -1$.

In order to prevent the point $\Delta^-(\mu_{12}, \mu_{23})$ from touching the contour of integration during these operations, we must deform this contour in the

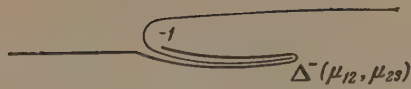


FIG. 4

way shown in Fig. 4. The integral along this contour will be an analytic function of μ_{12} and is identical with the integral (2) for $\mu_{12} < -\mu_{23}$. Thus the integral along the contour shown in Fig. 4 represents an analytic continuation of formula (2) into the region $\mu_{12} < -\mu_{23}$.

The integral along the contour C shown in Fig. 4 gives an anomalous addition to formula (2). This term can be reduced to an integral of the discontinuity of A_1 along the segment $(-1, \Delta^-(\mu_{12}, \mu_{23}))$ of the cut coming from the point $\Delta^-(\mu_{12}, \mu_{23})$. Let us consider, for example, the case where

$$\Delta^-(\mu_{12}, \mu_{23}) = \Delta_1(\mu_{13}).$$

In the region $-1 < \mu_{13} < \Delta_1(\mu_{13})$ we have

$$V^2 > K', \quad (25)$$

$$\sqrt{K'} < 0, \quad (26)$$

$$V < 0; \quad (27)$$

the inequality (25) follows from (6) and (7), and (26) is obvious from Fig. 4. In order to prove (27), we note that the quantity V does not vanish for $\mu_{12} + \mu_{23} < 0$ in the region $\mu_{24} > 1$, $-1 < \mu_{13} < \Delta_1(\mu_{13})$, and $V \rightarrow -\infty$ for $\mu_{24} \rightarrow \infty$, as is seen from formula (4).

It follows from (25) – (27) that $\Delta^-(\mu_{12}, \mu_{23})$ is a singular point for $\ln \xi_1$, and $\xi_1 < 0$ on the cut. On the other hand, $d\xi_1/d\mu_{13} > 0$ in the point $\mu_{13} = \Delta_1(\mu_{13})$, since

$$\begin{aligned} \frac{d\xi_1}{d\mu_{13}} \Big|_{\mu_{13}=\Delta_1} &= \frac{1}{\xi_2} \frac{d(\xi_1 \xi_2)}{d\mu_{13}} \Big|_{\mu_{13}=\Delta_1} \\ &= \frac{1}{\xi_2} \frac{d}{d\mu_{13}} [\lambda(\mu_{12}, \mu_{23}, \mu_{13}) \lambda(\mu_{14}, \mu_{34}, \mu_{13})] \Big|_{\mu_{13}=\Delta_1} \\ &= \frac{\lambda(\mu_{14}, \mu_{34}, \mu_{13})}{\xi_2} \frac{d}{d\mu_{13}} \lambda(\mu_{12}, \mu_{23}, \mu_{13}) \Big|_{\mu_{13}=\Delta_1} > 0. \end{aligned} \quad (28)$$

Hence $\text{Im } \xi_1 > 0$ near the point $\Delta_1(\mu_{13})$ if μ_{13} is in the upper half-plane, and $\text{Im } \xi_1 < 0$ if μ_{13} is in the lower half-plane. In other words, $\arg \xi_1 = \pi$ on the upper branch of the branch cut and $\arg \xi_2 = -\pi$ on the lower branch. The anomalous term $A_{an}(\mu_{ik})$ is therefore equal to

$$\begin{aligned} A_{an}(\mu_{ik}) &= -\frac{2\pi i}{\pi} \int_{-1}^{\Delta^-(\mu_{12}, \mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \\ \sqrt{K} &= i |\sqrt{K}|. \end{aligned} \quad (29)$$

Formula (29) is evidently also valid in the case when the point $\Delta^-(\mu_{12}, \mu_{23})$ is not the one with the

lowest value of all the $\Delta_i(\mu_{13})$, since the function in (29) is analytic in μ_{12} . For the same reason formula (29) maintains its form also when one continues with respect to μ_{23} . Hence formula (29) describes the anomalous term in the region $\mu_{24} > 1$ for

$$\theta_{12} + \theta_{23} > \pi, \quad \theta_{14} + \theta_{34} < \pi \quad (30)$$

It is seen from (29) that the anomalous addition to formula (2) is expressed in the form of an integral of a function which is formally identical with the Mandelstam function A_{12} , which in our case is equal to

$$A_{12}(\mu_{ik}) = 2\pi / |\sqrt{K}| \quad (31)$$

(see the following section).

3. ANALYTIC CONTINUATION INTO THE REGION $\mu_{24} < 1$

Let us now establish a dispersion representation for $A(\mu_{ik})$ in the region $\mu_{24} < 1$. Here we must distinguish between two cases:

$$1) \quad \theta_{12} + \theta_{23} + \theta_{14} + \theta_{34} < 2\pi, \quad (32)$$

$$2) \quad \theta_{12} + \theta_{23} + \theta_{14} + \theta_{34} > 2\pi. \quad (33)$$

In the first case Mandelstam's double representation is valid, as will be shown below. We shall call this case the normal case. In the second (anomalous) case there does not exist a double representation, since the function $A(\mu_{ik})$ has complex singularities.

1) Let us consider the analytic continuation in the normal case. If condition (32) is satisfied, we have, according to (14) and (15),

$$\Delta^-(\mu_{12}, \mu_{23}) = \Delta_1(\mu_{13}).$$

The expression under the integral sign in formula (29) for $A_{an}(\mu_{ik})$, regarded as a function of μ_{24} , has the singular points $\square_1(\mu_{13})$ and $\square_2(\mu_{13})$, which lie on the curve II. We must draw the cut between these points in such a way that $\sqrt{K} = i|\sqrt{K}|$ for $\mu_{24} > \square_2(\mu_{13})$. Then formula (29) can be written as a double dispersion integral

$$\frac{2\pi}{\pi^2} \int d\mu'_{13} d\mu'_{24} / [V\sqrt{K}(\mu'_{13} - \mu_{13})(\mu'_{24} - \mu_{24})] \quad (34)$$

in the region bounded by the curve II and the straight line $\mu_{13} = -1$.

The analytic continuation of the dispersion integral (2) in its general form has been carried out by Mandelstam. He obtains the double integral (34) in the region $\mu_{24} < \square_1(\mu_{13})$, $\mu_{13} < -1$. The sum of this integral and the double integral for A_{an} gives the double integral (34) in the region bounded by

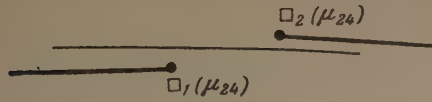


FIG. 5

the curve II. The double Mandelstam representation is therefore valid if condition (32) is satisfied. This result has been obtained earlier.^[1]

For real μ_{ik} in the region $\mu_{24} < 1$ we shall interpret the function $A(\mu_{ik})$ as the limit

$$A(\mu_{13} - i\delta, \mu_{24} - i\delta), \quad \delta \rightarrow +0. \quad (35)$$

With this definition, the function $A(\mu_{ik})$ is the scattering amplitude for the process

$$(p_{12}, p_{14}) \rightarrow (p_{23}, p_{34})$$

in the region $\mu_{13} > 1, \mu_{24} < -1$.

We shall show now that under condition (32) those points of the curve II for which $\mu_{13} < \mu_{13}^0, \mu_{24} < \mu_{24}^0$, where μ_{13}^0 and μ_{24}^0 are the tangent points of the curve II and the straight lines $\Delta_1(\mu_{24})$ and $\Delta_1(\mu_{13})$, are not singular points of the function $A(\mu_{ik})$ as defined by (35). For this purpose we write (2) in the form

$$A(\mu_{ik}) = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{1}{\sqrt{K}} \ln \frac{-\xi_1}{-\xi_2} \frac{d\mu_{13}'}{\mu_{13}' - \mu_{13}} - \frac{2\pi i}{\pi} \int_{-\infty}^{-1} \frac{d\mu_{13}'}{\sqrt{K}(\mu_{13}' - \mu_{13})}, \quad (36)$$

where

$$\sqrt{K} > 0 \text{ for } \mu_{24} < \square_1(\mu_{13}),$$

$$\sqrt{K} = i|\sqrt{K}| \text{ for } \mu_{24} > \square_1(\mu_{13});$$

$$\arg(-\xi_1) = \arg(-\xi_2) = 0, \text{ if } \mu_{24} < \square_1(\mu_{13}).$$

Curve II is not a singular curve for the first term on the right-hand side of (36). The second term is an integral along the upper branch of the cut coming from the point $\square_1(\mu_{24})$ and along the lower branch of the cut coming from the point $\square_2(\mu_{24})$ (see Fig. 5).

If the function $A(\mu_{ik})$ is defined according to (35) for real μ_{ik} , we can move the contour of integration into the upper half-plane in such a way that it does not go through the point $\square_1(\mu_{24})$, i.e., this point is actually not singular. It will become clear in the following that this result holds true also in the presence of anomalies.

The point $\square_2(\mu_{24})$ is a singular point of the second term on the right-hand side of (2) if $\square_2(\mu_{24}) < -1$, and of $A_{an}(\mu_{ik})$ if $\mu_{24} > \mu_{24}^0$, $\square_2(\mu_{24}) > -1$. Therefore the point $\square_2(\mu_{24})$ is a singular point of $A(\mu_{ik})$ for $\mu_{24} > \mu_{24}^0$. For $\mu_{24} < \mu_{24}^0$ the point $\square_2(\mu_{24}) > -1$ is not a singular

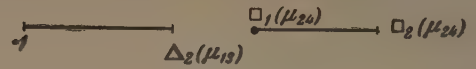


FIG. 6

point of $A(\mu_{ik})$ if $A(\mu_{ik})$ is defined according to (35), as can easily be verified.

Thus only those points of curve II are singular points of $A(\mu_{ik})$ which satisfy the conditions $\mu_{13} > \mu_{13}^0, \mu_{24} > \mu_{24}^0$.

2) Let us now consider the anomalous case.

The analytic continuation of formula (29) for $A_{an}(\mu_{ik})$ will be carried out in three steps: first into the region $\Delta_1(\mu_{24}) < \mu_{24} < 1$, then into the region $-1 < \mu_{24} < \Delta_1(\mu_{24})$, and finally into the region $\mu_{24} < -1$.

We first consider the analytic continuation into the region

$$\Delta_1(\mu_{24}) < \mu_{24} < 1.$$

Here two cases can occur:

$$a) \quad \theta_{12} + \theta_{23} < 2\pi - |\theta_{14} - \theta_{34}|, \quad (37)$$

$$b) \quad \theta_{12} + \theta_{23} > 2\pi - |\theta_{14} - \theta_{34}|. \quad (38)$$

It follows from (14) and (15) that in the first case

$$\Delta^-(\mu_{12}, \mu_{23}) = \Delta_2(\mu_{13}), \quad (39)$$

whereas in the second case

$$\Delta^-(\mu_{12}, \mu_{23}) = \Delta_3(\mu_{13}). \quad (40)$$

a) Let us consider the first case. As long as $\mu_{24} > \bar{\mu}_{24}$, where $\bar{\mu}_{24}$ is the tangent point of the straight line $\Delta_2(\mu_{13})$ and the curve V (see Fig. 2), formula (29) evidently represents a function which is analytic in μ_{12} with the condition that $\sqrt{K} = i|\sqrt{K}|$. The singular points of $\sqrt{K}$ in the μ_{13} plane and the cuts leading away from these points are shown in Fig. 6.

For the analytic continuation into the region $\mu_{24} < \bar{\mu}_{24}$ we shall assume that

$$\mu_{24} = \bar{\mu}_{24} - i\delta. \quad (41)$$

If $\delta > 0$ in (41), it can be easily shown that for $\mu_{24} > \bar{\mu}_{24}$ the point $\square_1(\mu_{24})$ lies in the lower and the point $\square_2(\mu_{24})$ in the upper half-plane of μ_{13} . For $\mu_{24} = \bar{\mu}_{24}$ the point $\square_1(\mu_{24})$ crosses the segment $(-1, \Delta_2(\mu_{13}))$ and moves then into the upper half-plane, taking the contour along (Fig. 7). For μ_{24}

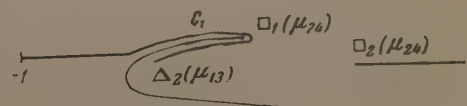


FIG. 7

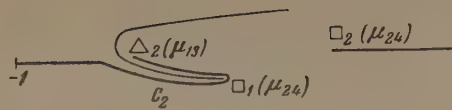


FIG. 8

$< \bar{\mu}_{24}$, the integration must therefore be carried out along the contour shown in Fig. 7.

The same thing happens if $\delta < 0$ in (41), only that now the point $\square_1(\mu_{24})$ moves from the upper into the lower half-plane (Fig. 8).

As long as $\mu_{24} > \Delta_2(\mu_{24})$, the contours C_1 and C_2 shown in Figs. 7 and 8 lie on the real axis and the integrals along them coincide. The point $\bar{\mu}_{24}$ is therefore not a singular point of $A(\mu_{ik})$.

The anomalous term in the region $\Delta_2(\mu_{24}) < \mu_{24} < 1$ has the form

$$A_{an} = -\frac{2\pi i}{\pi} \int_{-1}^{\Delta^-(\mu_{12}, \mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{4\pi i}{\pi} \theta(\bar{\mu}_{24} - \mu_{24}) \int_{\Delta^-(\mu_{12}, \mu_{23})}^{\square_1(\mu_{24})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (42)$$

where $\sqrt{K} = i|\sqrt{K}|$, and $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

For $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$ the contours C_1 and C_2 move into the complex plane and the integrals along them will no longer coincide. The point $\Delta_2(\mu_{24})$ is therefore a singular point for $A(\mu_{ik})$. It follows from (41), (14), and (15) that

$$\Delta_2(\mu_{24}) = \max[\Delta^-(\mu_{12}, \mu_{14}), \Delta^-(\mu_{23}, \mu_{34})].$$

We note that the singular point with respect to μ_{13} for $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$ is situated in the upper half-plane of μ_{13} , if $A(\mu_{ik})$ is defined according to (35). Thus the amplitude is an analytic function of s in the upper half-plane.

b) Let us now assume that condition (40) is satisfied. In this case the singular points $\square_1(\mu_{24})$ and $\square_2(\mu_{24})$ fall on the segment $(-1, \Delta^-(\mu_{12}, \mu_{23}))$ immediately after passing the point

$$\Delta_3(\mu_{24}) = \max[\Delta^-(\mu_{12}, \mu_{14}), \Delta^-(\mu_{23}, \mu_{34})].$$

Considerations completely analogous to those in case a) show that the point $\Delta_3(\mu_{24})$ is a singular point for $A(\mu_{ik})$, and after passing the point $\bar{\mu}_{24}$, where $\bar{\mu}_{24}$ is the tangent point of the curve V and the straight line $\Delta_2(\mu_{13})$, the contour of integration is deformed, just as in case a). The anomalous term in the region $\Delta_2(\mu_{24}) < \mu_{24} < 1$ has the form

$$A_{an} = -\frac{2\pi i}{\pi} \int_{-1}^{\Delta^-(\mu_{12}, \mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (43)$$

where $\sqrt{K}$ is defined in the following way:

For $\mu_{24} > \bar{\mu}_{24}$

$$\sqrt{K} = i|\sqrt{K}|, \text{ if } \mu_{13} \text{ is outside the interval } (\square_1(\mu_{24}), \square_2(\mu_{24})); \quad (44)$$

$$\sqrt{K} > 0, \text{ if } \square_2(\mu_{24}) < \mu_{13} < \square_1(\mu_{24}) \text{ and in (41) } \delta > 0; \quad (45)$$

$$\sqrt{K} < 0, \text{ if } \mu_{13} \text{ is inside the interval } (\square_1(\mu_{24}), \square_2(\mu_{24}))$$

$$\text{and in (41) } \delta < 0. \quad (46)$$

When $\mu_{24} < \bar{\mu}_{24}$ we have

$$\sqrt{K} = i|\sqrt{K}| \text{ for } \mu_{13} < \square_1(\mu_{24}), \quad (47)$$

$$\sqrt{K} = -i|\sqrt{K}| \text{ for } \mu_{13} > \square_1(\mu_{24}) \quad (48)$$

and when

$$\square_1(\mu_{24}) < \mu_{13} < \square_2(\mu_{24})$$

$\sqrt{K}$ is defined by (45) and (46). Here it turns out that only those points of the curve V are singularities of the function $A(\mu_{ik})$, defined according to (35), which satisfy the condition $\mu_{13} > \bar{\mu}_{13}$, $\mu_{24} > \bar{\mu}'_{24}$, where $\bar{\mu}_{13}$ and $\bar{\mu}'_{24}$ are the tangent points of the curve V and the straight lines $\Delta_3(\mu_{24})$ and $\Delta_2(\mu_{13})$.

As in the case considered above, $A(\mu_{ik})$ has a complex singularity in the upper half-plane as a function of μ_{13} , if $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$. However, the point

$$\Delta_2(\mu_{24}) = \min[\Delta^+(\mu_{12}, \mu_{14}), \Delta^+(\mu_{23}, \mu_{34})]$$

is not a singular point of $A(\mu_{ik})$ in this case. The anomalous term is in this region equal to the sum of the integral (29) and an integral along the contours C_1 or C_2 , which are shown in Figs. 7 and 8.

Let us now consider the analytic continuation into the region $-1 < \mu_{24} < \Delta_1(\mu_{24})$.

a) Let us assume that condition (22) is fulfilled. In this case the curve III touches the straight line $\Delta_1(\mu_{24})$. After passing the point $\Delta_1(\mu_{24})$ the contours of integration therefore assume the position shown in Fig. 9 for the case $\delta > 0$ in (41) and in Fig. 10 for the case $\delta < 0$. The anomalous term has the form

$$A_{an}(\mu_{ik}) = -\frac{2\pi i}{\pi} \int_{-1}^{\Delta^-(\mu_{12}, \mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{4\pi i}{\pi} \int_{\Delta^-(\mu_{12}, \mu_{23})}^{\square_2(\mu_{24})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (49)$$

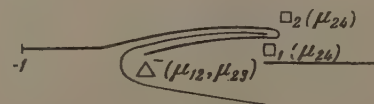


FIG. 9

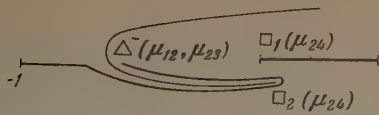


FIG. 10

where

$$\sqrt{K} = i|\sqrt{K}|, \quad \text{if } \mu_{13} < \square_1(\mu_{24}); \quad (50)$$

$$\sqrt{K} > 0 \quad \text{for } \mu_{13} > \square_1(\mu_{24}), \delta > 0;$$

$$\sqrt{K} < 0 \quad \text{for } \mu_{13} > \square_1(\mu_{24}), \delta < 0. \quad (51)$$

Considerations analogous to those above show that the point $\square_1(\mu_{24})$ is not a singularity for $A(\mu_{ik})$, defined according to (35). The point $\Delta_1(\mu_{24})$ is a singular point for $A(\mu_{ik})$ if

$$\theta_{12} + \theta_{14} > \pi, \quad \theta_{23} + \theta_{34} > \pi, \quad (52)$$

but is not a singular point in the opposite case. This can be seen by analytic continuation of the dispersion representation in μ_{24} with respect to the masses.

b) Let us now turn to the case

$$(\mu_{12} + \mu_{14})(\mu_{23} + \mu_{34}) < 0. \quad (53)$$

In this case the straight line $\Delta_1(\mu_{24})$ touches the curve II. Considerations analogous to those above show that the anomalous term will be

$$A_{an}(\mu_{ik}) = -\frac{4\pi i}{\pi} \theta(-1 - \square_2(\mu_{24})) \int_{\square_2(\mu_{24})}^{-1} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{-1}^{\Delta_1(\mu_{12}, \mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (54)$$

where $\sqrt{K} = -i|\sqrt{K}|$.

As in the above-mentioned cases, it can be shown that the point $\square_2(\mu_{24})$ is not a singular point of $A(\mu_{ik})$. The point $\Delta_1(\mu_{24})$ will evidently be a singular point, since it is in this point that the cut for $A(\mu_{ik})$ as a function of μ_{24} , coming from the points $\Delta_2(\mu_{24})$ or $\Delta_3(\mu_{24})$, ends.

Let us consider, finally, the analytic continuation into the region $\mu_{24} < -1$. It can be shown by the above-mentioned method that the anomalous term in this region is given by the formula

$$A_{an}(\mu_{ik}) = -\frac{4\pi i}{\pi} \theta(-1 - \square_1(\mu_{24})) \int_{\square_1(\mu_{24})}^{-1} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{-1}^{\Delta_1(\mu_{12}, \mu_{23})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (55)$$

where

$$\sqrt{K} = -i|\sqrt{K}|, \quad \text{if } \delta > 0 \text{ in (41),}$$

$$\sqrt{K} = i|\sqrt{K}| \quad \text{for } \delta < 0.$$

The point $\square_1(\mu_{24})$ is not a singular point of $A(\mu_{ik})$, as defined according to (35).

4. ANALYTIC CONTINUATION WITH RESPECT TO μ_{14} AND μ_{34}

The formulas for the anomalous term obtained in the preceding section are valid if condition (30) is satisfied. In order to determine the anomalous term for the case when the masses of all particles are large, i.e., when

$$\theta_{12} + \theta_{23} > \pi, \quad \theta_{14} + \theta_{34} > \pi, \quad (56)$$

we must carry out an analytic continuation of the dispersion representation for $A(\mu_{ik})$ with respect to μ_{14} and μ_{34} . Condition (56) implies again the inequality (21), so that the curve I touches the straight line $\Delta_1(\mu_{13})$.

The analytic continuation with respect to μ_{14} and μ_{34} into the region $1 < \mu_{24} < \bar{\mu}_{24}^0$, where $\bar{\mu}_{24}^0$ is the tangent point of the curve I and the straight line $\Delta_1(\mu_{13})$, is carried out in the same way as the analytic continuation with respect to μ_{12} and μ_{23} . The formula for the anomalous term in this region has the form

$$A_{an} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (57)$$

where

$$\beta = \max[\Delta_1^-(\mu_{12}, \mu_{23}), \Delta_1^-(\mu_{14}, \mu_{34})], \quad \sqrt{K} = i|\sqrt{K}|.$$

The anomalous terms for other values of μ_{24} can be obtained by analytic continuation with respect to μ_{24} .

In the region $\mu_{24} > \bar{\mu}_{24}^0$ the anomalous term is

$$A_{an} = -\frac{4\pi i}{\pi} \int_{\square_1(\mu_{24})}^{\Delta_1(\mu_{13})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (58)$$

where $\sqrt{K} = i|\sqrt{K}|$.

Therefore, only those points of the curve I are singular points of $A(\mu_{ik})$ which satisfy the condition $\mu_{24} > \bar{\mu}_{24}^0$.

In the region $\Delta_2(\mu_{24}) < \mu_{24} < 1$ the formula for the anomalous term with condition (37) has the form

$$A_{an} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{4\pi i}{\pi} \theta(\bar{\mu}_{24} - \mu_{24}) \int_{\beta}^{\square_1(\mu_{24})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (59)$$

where $\sqrt{K} = i|\sqrt{K}|$, and $\bar{\mu}_{24}$ is the tangent point of the curve V and the straight line $\Delta_2(\mu_{13})$.

If the condition (38) is satisfied, the anomalous term is equal to

$$A_{an} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad \sqrt{K} = i|\sqrt{K}|. \quad (60)$$

In the region $\Delta_2(\mu_{24}) < \mu_{24} < \Delta_1(\mu_{24})$ the anomalous term A_{an} is equal to the sum of expression (60) and an integral along the contours C_1 or C_2 , which are shown in Figs. 8 and 9. All considerations of the preceding section concerning the singular points $\Delta_2(\mu_{24})$ and $\square_2(\mu_{24})$ are also valid in this case.

In the region $-1 < \mu_{24} < \Delta_1(\mu_{24})$ we have, with condition (22),

$$A_{an} = -\frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{4\pi i}{\pi} \int_{\square_2(\mu_{24})}^{\Delta_2(\mu_{24})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (61)$$

where $\sqrt{K}$ is defined according to (50) and (51).

If condition (53) is fulfilled,

$$A_{an} = -\frac{4\pi i}{\pi} \theta(\Delta_1(\mu_{13}) - \square_2(\mu_{24})) \int_{\square_2(\mu_{24})}^{\Delta_1(\mu_{13})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (62)$$

where $\sqrt{K} = -i|\sqrt{K}|$.

If $\mu_{24} < -1$, then

$$A_{an} = -\frac{4\pi i}{\pi} \int_{\square_1(\mu_{24})}^{\Delta_1(\mu_{13})} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})} - \frac{2\pi i}{\pi} \int_{\Delta_1(\mu_{13})}^{\beta} \frac{d\mu'_{13}}{\sqrt{K}(\mu'_{13} - \mu_{13})}, \quad (63)$$

where $\sqrt{K}$ is defined according to (58).

In the case (56), as in the case (30), the curve II is not a singular curve for $A(\mu_{ik})$.

Thus, if (33) is satisfied, the anomalous additions to the dispersion representation (2) are given for $\mu_{24} > \Delta_4(\mu_{24})$ by formulas (29), (57), and (58), for $\Delta_2(\mu_{24}) < \mu_{24} < \Delta_4(\mu_{24})$ by formulas (42), (43), (59), and (60), for $-1 < \mu_{24} < \Delta_1(\mu_{24})$ by formulas (49), (54), (61), and (62), and for $\mu_{24} < -1$ by formulas (55) and (63).

In the region $\Delta_1(\mu_{24}) < \mu_{24} < \Delta_2(\mu_{24})$ the function $A_{an}(\mu_{ik})$ has a complex singularity in μ_{13} , which is situated in the upper half-plane. In this region the anomalous term is given by the sum of the integrals (43) or (60) and integrals along the contours C_1 or C_2 , shown in Figs. 7 and 8. In all cases the anomalous term is given by some integral of a function which is formally identical with the Mandelstam function A_{12} .

If condition (32) is satisfied, there exists a double Mandelstam representation in the form of the integral (34) over the region bounded by the curve II.

All the formulas above have been derived under the assumption that the squares of the masses of the external particles have an infinitesimal positive imaginary part. It turns out, however, that one obtains the same results if this small addition is taken to be negative. Therefore the values of the masses for which the anomalous additional terms appear are not singularities of $A(\mu_{ik})$.

¹J. Tarski, J. Math. Phys. **1**, 149 (1960), S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

²S. Mandelstam, Phys. Rev. Lett. **4**, 84 (1960).

³Gribov, Terent'ev, and Ter-Martirosyan, JETP **40**, 337 (1961), Soviet Phys. JETP **13**, 229 (1961).

⁴S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

⁵Karplus, Sommerfield, and Wichmann, Phys. Rev. **114**, 376 (1959).

⁶L. D. Landau, JETP **37**, 62 (1959), Soviet Phys. JETP **10**, 45 (1960).

THRESHOLD SINGULARITIES IN THE TOTAL PION SCATTERING CROSS SECTION OF PROTONS

N. P. KLEPIKOV, V. R. ROKITYANSKII, Yu. G. RUDOĬ, F. V. SAEVSKII, V. V. FEDOROV, and V. A. YUDIN

Institute of Nuclear Physics, Moscow State University

Submitted to JETP editor April 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 937-938 (September, 1961)

An analysis of the experimental data indicates the possible existence of singularities in the total cross section for the scattering of positive pions by protons near the thresholds for production of an additional pion.

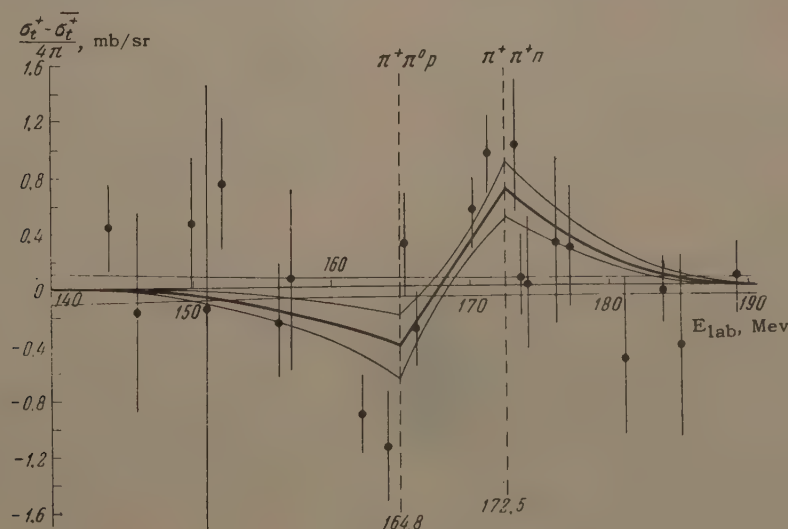
QUANTUM-MECHANICAL calculations^[1] show that the elastic scattering cross section can have singularities at the thresholds for inelastic reactions. This paper describes an attempt to find such singularities in the total cross section for the scattering of positive pions by protons. We consider the energy region from 140 to 190 Mev, which contains the thresholds for the reactions $\pi^+p \rightarrow \pi^+\pi^0p$ and $\pi^+p \rightarrow \pi^+\pi^+n$ at 164.8 and 172.5 Mev, respectively.

On the basis of more than 250 experimental cross sections for scattering of positive and negative pions, an interpolation formula with 15 parameters has been found^[2] which describes satisfactorily the total cross sections for interactions of pions with protons at all energies. In particular, the cross section in the region under consideration is essentially described by a Breit-Wigner resonance formula.

In order to study the threshold singularities, the values of the interpolation formula of^[2] were subtracted from all the experimental points in this

region (literature references for these are given in^[2]). The results of this subtraction are shown in Fig. 1. The error bars shown on Fig. 1 include the effect of errors in the average energy of the pion beams. The two fine lines nearly parallel to the energy level corresponding to the average cross section show the "error corridor" due to the general interpolation. The experimental points in the region under consideration are much more widely scattered about the average cross section than in neighboring intervals, and they are displaced from it by distances exceeding their errors. For the 22 points (the number of degrees of freedom is $f = 22$) we find $\chi^2/f = 2.12$, which gives a random scatter probability of merely 0.16%. It is also evident that the points lie mainly below the average cross section near the lower threshold and above it near the upper one.

Therefore we performed an additional interpolation of the total cross section in this region with functions of the form (σ and $\bar{\sigma}$ are in mb and energies are in Mev):



$$(\sigma - \bar{\sigma})/4\pi = a(E - 140)^2, \quad 140 < E < 164.8 \text{ Mev}$$

$$(\sigma - \bar{\sigma})/4\pi = (Aa + Bb) + (Ca + Db)E,$$

$$164.8 < E < 172.5 \text{ Mev}, \quad (\sigma - \bar{\sigma})/4\pi = b(E - 190)^2,$$

$$172.5 < E < 190 \text{ Mev},$$

where the coefficients $A = 1.38 \times 10^4$, $B = -6.56 \times 10^3$, $C = -79.9$, and $D = 39.8$ were determined from the condition that the curves match at the thresholds.

By minimizing χ^2 , the weighted sum of square deviations of the points from the line, we obtained $a = -7.1 \times 10^{-4}$ and $b = 2.14 \times 10^{-3}$. The resulting broken line is shown in Fig. 1 together with its error corridor as determined by the procedure described previously.^[3] Now $f = 20$ and $\chi^2/f = 1.37$; the probability of random scatter of the points about the broken line has increased 100 times. It is evident from Fig. 1 that the error corridors of the two interpolations do not overlap at the thresholds. This gives some basis for recommending that the cross sections in this region be remeasured with accuracy sufficient to show the presence and nature of the threshold pion-scattering singularities which are able to give additional information concerning the pion-pion interaction.

The foregoing possibility that threshold singularities are present in the cross section must be regarded with caution. An analogous calculation, with nonnormalized data, on the scattering of negative pions by protons^[4] gave a negative result. It

was not possible to fit the cross section near the $\pi^-p \rightarrow \pi^0\pi^0n$, $\pi^-p \rightarrow \pi^-\pi^0p$, and $\pi^-p \rightarrow \pi^+\pi^-n$ reaction thresholds at 160.7, 164.9, and 172.5 Mev, respectively, any better with a broken line with straight segments than with a parabola. The result was $\chi^2/f = 1.08$ with $f = 21$ for the best line with four segments, while the parabola gave $\chi^2/f = 1.06$ with $f = 23$. Therefore, only further experiments will allow us to ascertain the existence of threshold singularities in the cross section for scattering of positive pions on protons.

The authors are grateful to Professor Ya. A. Smorodinskii, S. N. Sokolov, A. I. Mukhin, V. G. Zinov, and Professor B. Pontecorvo for useful discussions.

¹A. I. Baz', JETP **33**, 923 (1957) and **36**, 1762 (1959), Soviet Phys. JETP **6**, 709 (1958) and **9**, 1256 (1959).

²Klepikov, Meshcheryakov, and Sokolov, preprint D-584, Joint Institute for Nuclear Research, 1960.

³N. P. Klepikov and S. N. Sokolov, preprint R-235, Joint Institute for Nuclear Research, 1958.

⁴Zinov, Konin, Korenchenko, and Pontecorvo, JETP **38**, 1708 (1960), Soviet Phys. JETP **11**, 1233 (1960).

Translated by M. Bolsterli

MOTION OF CHARGED QUASIPARTICLES IN A VARYING INHOMOGENEOUS ELECTRO-MAGNETIC FIELD

I. M. LIFSHITZ, A. A. SLUTSKIN, and V. M. NABUTOVSKII

Physico-Technical Institute, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor April 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 939-948 (September, 1961)

We consider the motion of charged quasiparticles in electric and magnetic fields varying slowly in time and space. We derive and investigate "averaged" equations of motion for various geometries of the equal-energy surfaces. Properties of the motion in the vicinity of saddle-points in $\mathbf{p}$ space are studied. It is found that for such points there arises a peculiar type of scattering which is not connected with the presence of a force center in configuration space. We calculate the probabilities of the particle arriving in various regions for different types of motion.

1. INTRODUCTION

ALL statistical, thermodynamical, and kinetic properties of metals and semiconductors are related to the dynamics of charged quasiparticle motion, the charged quasiparticles being the current carriers in such substances. The energy $\epsilon(\mathbf{p})$ of a quasiparticle is a complicated periodic function of the quasimomentum $\mathbf{p}$, the period being that of the reciprocal lattice multiplied by $2\pi\hbar$. In the ideal-gas approximation, which is good enough for describing most of the phenomena, the dynamics of the motion is determined by the dispersion law of $\epsilon(\mathbf{p})$, for which it is important to know the geometry of the isoenergetic surfaces. Anisotropy in $\epsilon(\mathbf{p})$ leads to several important peculiarities in the quasiparticle motion as compared with the motion of free electrons. These peculiarities are manifested in macroscopic properties of metals and semiconductors when the mean free path is much greater than lengths of the order of the trajectory in $\mathbf{r}$ space. This is a condition which is fulfilled for sufficiently low temperatures.

The equations of motion of noninteracting particles in electric and magnetic fields are the usual Lorentz equations

$$\dot{\mathbf{p}} = e\mathbf{E} + (e/c)[\mathbf{v}\mathbf{H}], \quad \mathbf{v} = \partial\epsilon/\partial\mathbf{p}, \quad (1)^*$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ the magnetic field, and $\mathbf{v}$ the velocity of the particle.

The motion of a quasiparticle in a homogeneous constant magnetic field is well understood. For this case the trajectory of the particle in $\mathbf{p}$ space

is given by $\epsilon = \text{const}$, $p_H = \text{const}$, where p_H is the component of the momentum parallel to the magnetic field. Further, the motion in $\mathbf{p}$ space and $\mathbf{r}$ space depends strongly on the topological properties of this trajectory curve $\epsilon = \text{const}$, $p_H = \text{const}$. For a closed trajectory the motion in $\mathbf{p}$ space is periodic with period $T_0 = m^*c/eH$, where m^* is the effective mass for the region enclosed by the trajectory;^[1] the motion in $\mathbf{r}$ space is unbounded only in the direction of the magnetic field. For an open trajectory, the motion in $\mathbf{r}$ space is unbounded also in directions perpendicular to the magnetic field.

If the magnetic field is allowed to vary in space and time, and also if there exists an electric field parallel to $\mathbf{H}$, the energy and the component of the momentum along the magnetic field are no longer integrals of the motion. The nonconservation of ϵ and p_H , as well as significant anisotropies in the dispersion law lead to unique peculiarities in the quasiparticle motion.

The present article is a study of quasiparticle motion in electric and magnetic fields varying "slowly" in time and space. Such fields satisfy the conditions

$$\gamma_t = T_0/t_0 \ll 1, \quad \gamma_L = R_0/L \ll 1, \quad \gamma_E = cE/vH \ll 1. \quad (2)$$

Here t_0 and L are, respectively, a characteristic time and length for the variation of the electromagnetic field, and R_0 is the radius of curvature of the trajectory in $\mathbf{r}$ space. In practice these conditions are fulfilled up to very large field gradients and frequencies (for instance for H of the order of 10^3 oersteds, gradients of the order of

* $[\mathbf{v}\mathbf{H}] = \mathbf{v} \times \mathbf{H}$.

$\nabla H/H \sim 10^2 \text{ cm}^{-1}$ and frequencies of the order of $\omega_0 \sim 10^8 \text{ sec}^{-1}$ are permissible). Conditions (2) mean that the true motion of the particle can be treated as the sum of two motions, namely a smooth variation of the "averaged" quantities $\mathbf{R} = \bar{\mathbf{r}}$, $P_H = \bar{p}_H$, and $\mathcal{E} = \bar{\epsilon}$, and rapid oscillations which depend on $\mathbf{R}$, P_H , and $\mathcal{E}$ as parameters.

For the closed-trajectory case, the motion in $\mathbf{p}$ space can be thought of as a combination of the drift, rotation, and deformation of the "current leaf," or of the curve $\epsilon = \mathcal{E}(t)$, $p_H = P_H(t)$ along which the particle keeps rapidly rotating.

If the equal-energy surface is not everywhere convex, the particle will be "scattered" by saddle points of the surface. This peculiar scattering does not depend on the existence of any force center in $\mathbf{r}$ space, but is related to the fact that a saddle point is a singular point (stationary point) for motion of a quasiparticle in a constant and homogeneous magnetic field. When dealing with an electromagnetic field satisfying (2), this point divides $\mathbf{p}$ space into several regions of essentially different types of motion. When the $\epsilon = \mathcal{E}(t)$, $p_H = P_H(t)$ surface passes through such a singular point, the type of motion of the particle changes abruptly, and the region it ends up in depends on the exact initial conditions. In our discussion it is the probability for scattering into these regions which is of physical interest. We shall calculate these probabilities for various transitions from one type of motion to another.

2. CLOSED TRAJECTORIES IN $\mathbf{p}$ SPACE

In studying the motion of quasiparticles in fields satisfying (2), we shall use the coordinates $\mathbf{r}$, p_H , ϵ , and τ (where τ is the angle variable which defines the position of the particle on the trajectory given by $\epsilon = \text{const}$, $p_H = \text{const}$). We shall write the functions $\mathbf{r}(t)$, $p_H(t)$, and $\epsilon(t)$ in the form

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{R}(t) + \boldsymbol{\rho}(t), & p_H(t) &= P_H(t) + \tilde{p}_H(t), \\ \epsilon(t) &= \mathcal{E}(t) + \tilde{\epsilon}(t), \end{aligned} \quad (3)$$

where the "averaged" variables $\mathbf{R} = \bar{\mathbf{r}}$, $P_H = \bar{p}_H$, and $\mathcal{E} = \bar{\epsilon}$ define the smooth motion of the particle;* $\mathbf{R}$ can then be thought of as the coordinate of the "center of the orbit," while P_H , $\mathcal{E}$, and $\xi(\mathbf{R}, t) = H(\mathbf{R}, t)/H(\mathbf{R}, t)$ determine the position of the "current leaf" in $\mathbf{p}$ space.

The period $T_0(t) = eH/m^*c$ is a function of the "instantaneous" variables ϵ , p_H , and ξ . The func-

*We shall henceforth define $\bar{\mathbf{X}}$ by

$$\frac{1}{T_0(t)} \int_t^{t+T_0(t)} X(t') dt' \quad (3a)$$

tions $\boldsymbol{\rho}$, $\tilde{p}_H$, and $\tilde{\epsilon}$ are rapidly oscillating increments, and they satisfy the relations $|\boldsymbol{\rho}| \sim R_0$, $\tilde{p}_H a/\hbar \sim \gamma \ll 1$ (where a is the period of the lattice), and $\tilde{\epsilon}/\epsilon \sim \gamma \ll 1$. Henceforth we shall devote our attention to the time variation of the "averaged" variables and derive their equations of motion.

We first write the exact equations of motion in terms of $\mathbf{r}$, p_H , and ϵ .

From (1) we obtain

$$\begin{aligned} \dot{\mathbf{p}}_H &= \mathbf{p}_\perp (\mathbf{v} \nabla) \xi + \mathbf{p}_\perp \partial \xi / \partial t + e E_\xi, \\ \dot{\epsilon} &= e E v, \quad \dot{\mathbf{r}} = \mathbf{v}_\perp + v_\xi \xi, \end{aligned} \quad (4)$$

where $\mathbf{p}_\perp$ and $\mathbf{v}_\perp$ are the projections of the momentum and velocity, respectively, on a plane perpendicular to the magnetic field.

The variables $\mathbf{R}$, P_H , and $\mathcal{E}$ satisfy the relation $\dot{\bar{\mathbf{X}}} = \bar{\mathbf{X}}(1 + O(\gamma))$. In averaging (4) the integration over the true time t' can be replaced, accurate to terms of order γ , by integration over the angle variable* τ along the section $\epsilon = \mathcal{E}$, $p_H = P_H$, $\xi = \xi(\mathbf{r}, t)$. For this integration $d\tau = c dp_\perp / e H v_\perp$, where $dp_\perp$ is the element of arc length of the curve $\epsilon = \mathcal{E}$, $p_H = P_H$. As is well known, such averaging gives $\bar{v}_\perp = 0$, $\bar{p}_x \bar{v}_y = \bar{p}_y \bar{v}_x = 0$, and $\bar{p}_x \bar{v}_x = \bar{p}_y \bar{v}_y = S/2\pi m^*$, where $S = S(p_H, \epsilon, \xi)$ is the area enclosed by the intersection of the surface $\epsilon = \mathcal{E}$ in the plane $p_H = P_H$, and $m^* = (2\pi)^{-1} \partial S / \partial \epsilon$.

Noting that $\mathbf{E}(\mathbf{R} + \boldsymbol{\rho}) = \mathbf{E}(\mathbf{R}) + (\boldsymbol{\rho} \cdot \nabla) \mathbf{E}$, and using the equation $\nabla \times \mathbf{E} = -c^{-1} \partial \mathbf{H} / \partial t$, we proceed to obtain the averages and arrive finally, after several relatively simple operations, to the set of equations

$$\begin{aligned} \dot{P}_H &= \frac{H}{2\pi m^*} (\xi \nabla) \frac{S}{H} + \mathbf{p}_\perp^0 \frac{\partial \xi}{\partial t} + e E_\xi, \\ \dot{\mathcal{E}} &= e E_\xi v_\xi^0 + M \partial H / \partial t, \quad \dot{\mathbf{R}} = v_\xi^0 \xi, \end{aligned} \quad (5)$$

where $\mathbf{M} = (e/2c) \mathbf{r} \times \mathbf{v} - (e/2c) \mathbf{R} \times \mathbf{v}$ is the mean magnetic moment of the particle about the orbital center $\mathbf{R}$, and

$$x^0 = \frac{1}{T} \int_0^T x d\tau.$$

The right sides of Eqs. (5) contain functions of P_H , $\mathcal{E}$, and $\mathbf{R}$ only. From Eqs. (5) and with the aid of the relations

$$v_\xi^0 = -\frac{1}{2\pi m^*} \frac{\partial S}{\partial p_H}, \quad \frac{\partial S}{\partial \xi} = \oint \frac{\mathbf{p}_\perp v_\xi}{v_\perp} dp_\perp$$

we can show easily that $J = S(P_H, \mathcal{E}, \xi)/H(\mathbf{R}, t)$ is an integral of the motion. Thus S/H is an adiabatic invariant not only for free electrons, but also

*This variable has often been used in studying the motion of quasiparticles in constant magnetic fields.

for charged quasiparticles with arbitrary dispersion law.*

Another important property of motion in a slowly space and time varying electromagnetic field is that the velocity of the orbital center is along the magnetic field.

The existence of the S/H integral of the motion allows one to introduce an important simplification into the discussion of (5). Let us consider some special cases.

1. Inhomogeneous magnetic field constant in time. For this case the orbital center $\mathbf{R}$ moves along a line of force of the magnetic field. If l is the arc length along such a line of force, the equations of motion can be written

$$\mathcal{E} = \text{const}, \quad S(P_H, \mathcal{E}, \xi(l))/H(l) = \text{const}, \\ \dot{l} = v_{\xi}^0(P_H, \mathcal{E}, \xi(l)).$$

2. Electric field parallel to a magnetic field ($\mathbf{E} = \text{const}$, $\mathbf{H} = \text{const}$). The equations of motion become

$$\dot{P}_H = eE, \quad S(P_H, \mathcal{E}) = \text{const}, \quad \dot{l} = v_{\xi}^0(P_H, \mathcal{E}).$$

For both the above cases the equations can be reduced to quadratures.

3. A varying magnetic field $\mathbf{H} = \mathbf{H}(t)$. Recalling that the electric field induced by the variation of the magnetic field is not energy conserving, we arrive at the following set of equations:

$$\dot{P}_H = p_{\perp}^0 \partial \xi / \partial t, \quad J(P_H, \mathcal{E}, t) = \text{const}, \\ \dot{\mathbf{R}} = v_{\xi}^0(P_H, \mathcal{E}, t) \xi(t).$$

3. OPEN PERIODIC TRAJECTORIES

If the $\epsilon = \text{const}$, $p_H = \text{const}$ curves are open, the averaged equations of motion are derived quite differently for the two cases of periodic and aperiodic curves. An $\epsilon = \text{const}$, $p_H = \text{const}$ curve is periodic if the direction in which it is open is parallel to some reciprocal lattice vector $\mathbf{B}$; then $\xi \perp \mathbf{B}$. If an open periodic trajectory occurs for at least one direction of $\xi \perp \mathbf{B}$, then it is easily shown that it will occur also for any cross section whose normal lies within some angle φ ($0 < \varphi \leq 2\pi$) bounded from above and below.

One must distinguish between two types of surfaces on which open periodic trajectories occur: (a) surfaces for which there exists a one-dimensional set (an angle equal to 2π bounded from above and below) of normal directions leading to open curves; in this case all the open trajectories are periodic; (b) surfaces for which there exists

a two-dimensional set (solid angle) of normal directions leading to open trajectories; in this case only rational trajectories (with $\xi \perp \mathbf{B}$) are periodic.

From the above discussion it is clear that for motion in a plane magnetic field such that $\xi(\mathbf{R}, t) \perp \mathbf{B}$, the $\epsilon = \epsilon(t)$, $p_H = p_H(t)$ curves remain periodic. In dealing with motion on periodic trajectories, we introduce the following set of coordinates: ξ is the unit vector in the direction of the magnetic field, $\mathbf{e}_1$ is the unit vector in the direction along which the curve is open, and $\mathbf{e}_2 = \xi \times \mathbf{e}_1$. In a plane field, $\mathbf{e}_1$ remains constant, and $\xi \parallel \mathbf{e}_2$. The averaged equations of motion can be derived in the same way as for the closed trajectories. By an averaged quantity $\bar{x}$ (where x may be any function of the coordinates $\mathbf{r}$, p_H , and ϵ) we shall now understand an average as defined in Eq. (3a) with $T_0(t)$ equal to the time of flight in passing through an elementary cell of the reciprocal lattice.

The averaged equations of motion are

$$\dot{P}_H = \frac{H}{2\pi m^*} (\xi \nabla) \frac{S}{H} + p_{\perp}^0 \frac{\partial \xi}{\partial t} + eE_{\xi}, \\ \dot{\mathcal{E}} = e(E_{\xi} v_{\xi}^0 + E_2 v_2^0) + M \partial H / \partial t, \\ \dot{\mathbf{R}} = v_{\xi}^0 \xi + v_2^0 \mathbf{e}_2; \\ S = \int_0^B p_2 dp_1, \quad 2\pi m^* = \frac{\partial S}{\partial \epsilon}, \quad v_2^0 = \frac{B}{2\pi m^*}, \quad (6)$$

where M is defined in the same way as in Eqs. (5).

The main difference between motion on open trajectories and motion on closed ones is that in the present case $v_2^0 \neq 0$, which means that the average velocity is not directed along a line of force of the magnetic field. This means that $J = S/H$ is not conserved. The equation for J now becomes

$$\dot{J} = \frac{B}{H} (eE_2 - p_2^0 \frac{\dot{H}}{H}) + \frac{B}{2\pi m^*} (\mathbf{e}_2 \nabla) J. \quad (7)$$

We note that in the case of an electric field $\mathbf{E} \parallel \mathbf{H}$ (with $\mathbf{E} = \text{const}$, $\mathbf{H} = \text{const}$), J is an adiabatic invariant, as is the case for closed trajectories.

4. OPEN APERIODIC TRAJECTORIES

When dealing with open aperiodic trajectories, the average must be taken over a time interval T such that $T_0 \ll T \ll t_0$, where T_0 is a time of the order of the time of flight of a quasiparticle through an elementary cell ($T_0 \sim \hbar / eHv_a$). When averaging bounded quantities defined on an $\epsilon = \text{const}$, $p_H = \text{const}$ curve, the integration may be extended over the entire cross section, since the difference

$$\lim_{T' \rightarrow \infty} \frac{1}{2T'} \int_{-T'}^{T'} x \, d\tau - \frac{1}{T} \int_t^{t+T} x \, dt'$$

*Assuming that the trajectory $\epsilon = \text{const}$, $p_H = \text{const}$ is closed.

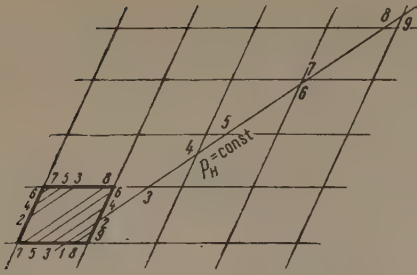


FIG. 1. One of the crystallographic surfaces of the reciprocal lattice, intersected by a $p_H = \text{const}$ surface. Equal numbers denote equivalent points.

is a rapidly oscillating expression with period of the order of xT_0/T (the variable τ is defined in the same way as for the case of closed trajectories

In calculating the mean value of such a quantity, the integration along an open trajectory passing through an infinite set of elementary cells can be replaced by a sum of integrals over equivalent segments within a single cell (Fig. 1). In the case of an aperiodic trajectory, these segments are dense and uniformly distributed in the cell. It follows then that $\bar{x}$ is independent of p_H . The p_H -dependence of averages can also be neglected for periodic trajectories with large period ($B \gg \hbar/a$). This independence of p_H makes it possible to drop this variable from our considerations, and to reduce the number of equations of motion; then these become

$$\begin{aligned} \dot{\mathbf{R}} &= v_{\xi}^0(\mathcal{E}, \xi) \xi(\mathbf{R}, t) + v_2^0(\mathcal{E}, \xi) \mathbf{e}_2(\mathbf{R}, t), \\ \dot{\mathcal{E}} &= e(E_{\xi} v_{\xi}^0 + E_2 v_2^0) + M \partial \mathbf{H} / \partial t, \end{aligned} \quad (8)$$

where $\mathbf{e}_2$ and $\mathbf{e}_1$ are defined as in the periodic case, and $M_i = \epsilon_{ikl} T_{kl} / 2c$.

The asymmetric tensor T_{kl} is given in terms of integrals along the curve $\epsilon = \mathcal{E}$, $\xi = \xi(\mathbf{R}, t)$ by the equations

$$\begin{aligned} T_{12} &= -(c/eH) \overline{p_2 v_2}, \quad T_{13} = -(c/eH) \overline{p_2 v_3}; \\ T_{23} &= -\lim_{T' \rightarrow \infty} \frac{1}{2T'} \int_{-T'}^{T'} d\tau' \tilde{v}_2 \int_{-T'}^{\tau'} \tilde{v}_3 d\tau'', \quad \tilde{x} = x - \bar{x}. \end{aligned}$$

For constant electric and magnetic fields, the integration reduces to quadratures:

$$\frac{1}{e} \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{E_{\xi} v_{\xi}^0(\mathcal{E}') + E_2 v_2^0(\mathcal{E}')} = t - t_0.$$

If the function $\mathcal{E}(t)$ is known, the first of Eqs. (8) can be integrated.

During its motion a particle may enter onto a periodic trajectory with period $B \sim \hbar/a$. In this case the p_H -dependence of $\bar{x}$ cannot be neglected. In the general case of a nonplane field [with $\xi(\mathbf{R}, t)$ not perpendicular to $\mathbf{B}$], a particle is in the neighborhood of such a trajectory for a time

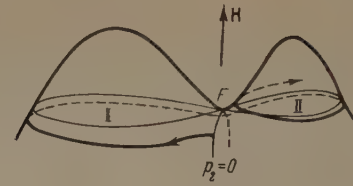


FIG. 2

of the order of $T_0 t_0 / T$. This means that one can use Eqs. (8) for times $\gtrsim t_0$.

5. SCATTERING ON A SINGULAR POINT

Scattering will occur in a slowly varying field if the motion undergoes transition from one type to another. Regions in $\mathbf{p}$ space with a different type of motion are separated from each other by segments of self-intersecting trajectories formed by the intersection of an $\epsilon = \text{const}$ surface and a plane tangent to an isoenergetic surface at a hyperbolic point. Classically such points are stationary points for motion in a homogeneous and constant magnetic field. The period of motion diverges logarithmically as $p_H \rightarrow f_H$ (where $f_H = \xi \cdot \mathbf{f}$, and $\mathbf{f}$ is the $\mathbf{p}$ -space position vector of the singular point).

It is easiest to understand the essentials of such scattering in terms of the example of a weakly inhomogeneous time-constant magnetic field which has at least the one straight line of force $\mathbf{r} = \mathbf{r}_0 + l\xi_0$. Let F be a saddle point on the surface $\epsilon = \epsilon_0$, such that the normal to the surface at this point is parallel to ξ_0 . The intersection of $\epsilon = \epsilon_0$ with $p_H = f_H$ is a figure-eight whose intersection point is at F . If, under the motion in $\mathbf{p}$ space, the "current leaf" $\epsilon = \epsilon_0$, $p_H = p_H(t)$, $\xi = \xi_0$ contacts the surface at the saddle point, it gets broken up into two "current leaves" in the two regions I and II which are separated by the singular point (Fig. 2). The types of motion are quite different in these two regions. Whether the particle enters region I or region II depends on the exact ("microscopic") initial conditions. The "microscopic" initial conditions are distributed so that each macroscopic energy-surface element determined by the averaged coordinates contains points from which the particle can enter region I as well as region II. From this point of view we can treat the entrance into either of these regions as a random process, and thus speak of the "scattering" of particles in the region of a singular point; then the probabilities w_1 and w_2 for scattering into each of the two regions have well defined values.

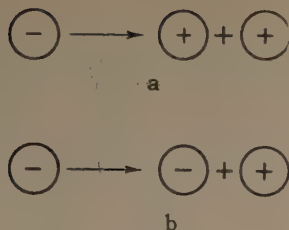


FIG. 3

To find w_1 and w_2 let us consider a classical ensemble of particles whose distribution is given in terms of some directional parameter to be determined later. On its last pass before entering into region I or II, every particle crosses the principal curvature line passing through F (for line $p_2 = 0$ in Fig. 2) for some value of $p_H(0)$. Having gone fully around one of the loops of the figure-eight, the particle finds itself again in the region of the self-intersection point. Then, depending on the sign of $p_H(t) - f_H$, the particle at this instant enters region I or II; the value of $p_H(t) - f_H$ is determined uniquely by $p_H(0)$ at the point where it was last intersected by the trajectory of the particle. It is seen from this that $p_H(0)$ is a convenient impact parameter to use. Let regions I and II correspond to intervals δ_I and δ_{II} of values of $p_H(0)$, and these will then determine the probabilities for entering these regions. The scattering probabilities w_1 and w_2 , i.e., the relative number of particles entering regions I and II, respectively, will be proportional to the flux of particles across δ_I and δ_{II} . For a sufficiently smooth distribution function these are proportional, to lowest order in γ , to the intervals themselves.

These intervals can be obtained from the relation

$$p_H(t) = p_H(0) + \int_0^t \dot{p}_H dt'.$$

From this we obtain

$$\delta_I = \int_0^{T_1} \dot{p}_H dt', \quad \delta_{II} = \int_{T_1}^{T_2} \dot{p}_H dt'.$$

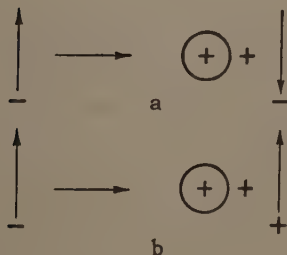


FIG. 4

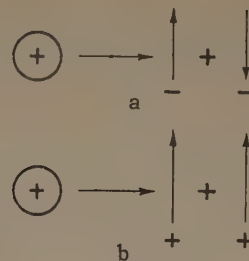


FIG. 5

(Here T_1 is the time it takes to go around the I loop of the figure-eight, and T_2 is the time it takes to go around the entire figure-eight; $T_{1,2} \sim T_0 \ln \gamma$.)

The largest contribution to the variation of p_H during the time it takes to go around the figure-eight is given by those parts of the trajectory which are far from the singular point; in other words, the total change in p_H is of order $\gamma \hbar/a$. This is because at the singular point itself (for a field whose direction is constant) $\dot{p}_H = 0$, and the contribution to the integral from points close to the singular point is of order $\gamma^2 \ln \gamma$. [We bear in mind the fact that the time it takes to go around depends logarithmically on $p_H(0) - f_H$.] It then follows that to first order in γ the integral expressions for δ_I and δ_{II} can be written

$$\delta_I = \oint_I \dot{p}_H d\tau, \quad \delta_{II} = \oint_{II} \dot{p}_H d\tau,$$

where the integrals are taken, respectively, over the I and II loops of the figure-eight (and τ is the angle variable introduced earlier). Using the first of Eqs. (5) and recalling that the field has a straight line of force, we obtain

$$w_1/w_2 = \delta_I/\delta_{II} = S_1/S_2, \quad (9)$$

where S_1 and S_2 are the areas enclosed by the loops of the figure-eight. From (9) we arrive at

$$w_1 = S_1/(S_1 + S_2), \quad w_2 = S_2/(S_1 + S_2).$$

In general there exist several types of transition from one kind of motion to another. These are indicated schematically in Figs. 3–5. The right sides of these figures contain symbols corresponding to the two regions into which the scattering takes place. A circle indicates a closed trajectory, while an arrow indicates an open periodic one, and the direction of the arrow indicates



FIG. 6



FIG. 7

the direction of motion along this periodic trajectory. The kinds of equal-energy surfaces which correspond to these transitions are shown in Figs. 6 – 8 by means of the $p_H = \text{const}$ contour lines. On these figures the self-intersection point is denoted by the letter F. The heavy lines denote self-intersecting curves. The arrows show the direction of motion along the trajectories. Figure 6 corresponds to diagrams 3a and b, Fig. 7 to diagrams 4a and 5a, and Fig. 8 to diagrams 4b and 5b. The signs in Figs. 3 – 5 give the sign of $p_H - f_H$ in the different regions.*

In an arbitrary field satisfying conditions (2), the singular point $f(\mathcal{E}, \xi)$ may move in $\mathbf{p}$ space (unlike the case considered above). Then the criterion for entering regions of different types of motion is the sign of $\Delta(t) = p_H(t) - f_H(t)$ [where $f_H(t) = f(t) \cdot \xi(t)$] when the particle is in the neighborhood of the singular point. Then the intervals δ_I and δ_{II} of $\Delta(0)$, which determine whether the particle enters regions I and II, are given for cases 3a, 4a, and 5a by

$$\delta_I = \left| \int_0^{T_1} (\dot{p}_H - \dot{f}_H) dt' \right|, \quad \delta_{II} = \left| \int_{T_1}^{T_2} (\dot{p}_H - \dot{f}_H) dt' \right|,$$

and for cases 3b, 4b, and 5b by

$$\delta_I = \left| \int_0^{T_2} (\dot{p}_H - \dot{f}_H) dt' \right|, \quad \delta_{II} = \left| \int_{T_1}^{T_2} (\dot{p}_H - \dot{f}_H) dt' \right|.$$

Here T_1 and T_2 are the first and second times when the particle is in the neighborhood of the singular points; as before, $T_{1,2} \sim T_0 \ln \gamma$.

Up to terms of order γ the expression for δ_I and δ_{II} can be written

$$\delta_I = \frac{c}{eH} \left| \int_{L_1} (\dot{p}_H - \dot{f}_H) \frac{dp_l}{v_\perp} \right|, \quad \delta_{II} = \frac{c}{eH} \left| \int_{L_2} (\dot{p}_H - \dot{f}_H) \frac{dp_l}{v_\perp} \right|, \quad (10)$$

where the contours L_1 and L_2 are the segments of the self-intersecting trajectories which bound the regions into which the particle may be scattered. (In the case of an open periodic trajectory, the integral is taken over one period.) Equations (10) hold for all types of transitions. It should be noted that in cases 3b, 4b, and 5b, scattering can occur



FIG. 8

only if the signs of $\int_{T_1}^{T_2} \dot{\Delta} dt'$ and $\int_0^{T_2} \dot{\Delta} dt'$ are opposite, while in cases 3a, 4a, and 5a, these signs must be the same.

To derive the formulas for the scattering probabilities in cases 3a and 3b, we make use of the adiabatic invariance of $J = S(P_H, \mathcal{E}, \xi)/H(R, t)$. Consider $S_{cr}^{1,2}(f_H(\mathcal{E}, \xi), \mathcal{E}, \xi)$ (S_{cr}^1 and S_{cr}^2 are the areas of the loops bounding the regions into which scattering takes place*). Using the fact $J_{1,2} = \text{const}$ to terms of order $\gamma \ln \gamma$, we obtain

$$T_{1,2} j_{cr}^{1,2} = T^{1,2} (j_{cr}^{1,2} - j^{1,2}) = (\partial S_{1,2} / \partial P_H) \bar{\Delta} T_{1,2}.$$

From this we obtain

$$\omega_1 / \omega_2 = \delta_I / \delta_{II} = j_{cr}^{(1)} / j_{cr}^{(2)} \Big|_{p_H=f_H}. \quad (11)$$

In deriving (11) we make use of the fact

$$\lim_{p_H \rightarrow f_H} \frac{T_1 \partial S_1 / \partial p_H}{T_2 \partial S_2 / \partial p_H} = 1.$$

Let us consider this equation for some special cases. In a time-constant weakly inhomogeneous magnetic field,

$$\frac{\omega_1}{\omega_2} = \frac{\partial}{\partial l} \left(\frac{S_1}{H} \right) / \frac{\partial}{\partial l} \left(\frac{S_2}{H} \right), \quad (12)$$

where l is the length of a line of force, and S_1 and S_2 are the areas of each of the loops on the $\epsilon = \text{const}$ surface defined by the intersection with the plane passing through the singular point and perpendicular to $\xi(l)$. For a straight line of force Eq. (11) goes over into (9).

In parallel electric and magnetic fields ($\mathbf{E} = \text{const}$, $\mathbf{H} = \text{const}$) we obtain

$$\frac{\omega_1}{\omega_2} = \frac{d}{dp_H} S_1(p_H, \epsilon_{cr}(p_H)) / \frac{d}{dp_H} S_2(p_H, \epsilon_{cr}(p_H)) \Big|_{p_H=f_H} \quad (13)$$

Transitions of types 4 and 5 differ topologically from those of type 3, since the former involve open periodic trajectories. Nevertheless in these cases also the formulas for the transition probabilities are obtained with the aid of the equation for $J = S/H$ (for a periodic trajectory $S = \int_0^B p_2 dp_1$, and the in-

*Six other types of transitions, essentially the same as those of Figs. 3–5, are obtained by changing the signs.

*In case 3b one of these loops is the entire figure-eight.

tegration is taken over one period). Using Eqs. (7) and proceeding similarly as above, we obtain the following expressions. For transitions of type 4a and b

$$\frac{w_1}{w_2} = \frac{\delta_1}{\delta_{11}} = j_{\text{cr}}^{(1)} / \left\{ j_{\text{cr}}^{(2)} - B \left(eE_2 - f_2 \frac{\dot{H}}{H} \right) + \frac{f_{\perp} (\partial \xi / \partial x_2) B v_f}{2\pi H} \right\}; \quad (14)$$

for transitions of type 5a and b

$$\frac{w_1}{w_2} = \frac{\delta_1}{\delta_{11}} = \frac{j_{\text{cr}}^{(1)} - B (eE_2 - f_2 \dot{H} / H) + f_{\perp} (\partial \xi / \partial x_2) v_f B / 2\pi H}{j_{\text{cr}}^{(2)} - B (eE_2 - f_2 \dot{H} / H) + f_{\perp} (\partial \xi / \partial x_2) v_f B / 2\pi H}, \quad (15)$$

where v_f is the component of the velocity at the singular point in the direction of the magnetic field.

¹I. M. Lifshitz and M. I. Kaganov, Usp. Fiz. Nauk **69**, 419 (1959), Soviet Phys.-Uspekhi **2**, 831 (1960).

Translated by E. J. Saletan
162

PRODUCTION OF LEPTON PARTICLE PAIRS ON A COULOMB CENTER

M. A. KOZHUSHNER and E. P. SHABALIN

Institute of Theoretical and Experimental Physics, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 949-953 (September, 1961)

The cross sections for the processes $\nu + A \rightarrow \nu + A + \mu^+ + \mu^-$ and $\mu + A \rightarrow \mu + A + \nu + \bar{\nu}$ occurring in the Coulomb field of the nucleus are calculated. It is shown that the Weizsäcker-Williams method can be applied to perform the calculations in the case under consideration.

THE present article is devoted to a study of the behavior of the cross section of production of lepton pairs when high-energy leptons are scattered in a Coulomb field. We have examined the processes

$$A + \nu \rightarrow A + \nu + \mu^+(e^+) + \mu^-(e^-), \quad (1)$$

$$A + \mu(e) \rightarrow A + \mu(e) + \nu + \bar{\nu}. \quad (2)$$

We used in the calculations a method analogous to that of Weizsäcker and Williams^[1] (see also^[2]). This method enables us to obtain the cross sections of the processes (1) and (2), if the respective cross sections are known for the reactions

$$\gamma + \nu \rightarrow \nu + \mu^+(e^+) + \mu^-(e^-), \quad (1')$$

$$\gamma + \mu(e) \rightarrow \mu(e) + \nu + \bar{\nu}. \quad (2')$$

We have therefore first calculated the cross sections of processes (1') and (2'). (The cross section of the process (2') was calculated earlier,^[3] but the numerical coefficients in the formula of^[3] are incorrect.)

In Sec. 2 we discuss the possibility of applying a method similar to that of Weizsäcker and Williams to the problem under consideration; our attention was called to this possibility by I. Ya. Pomeranchuk.

In the last section are contained results on the cross sections of production of $\mu^+\mu^-$ and $\nu\bar{\nu}$ pairs in scattering of a neutrino and muon respectively in the field of the nucleus.

1. CROSS SECTION OF PRODUCTION OF LEPTON PAIRS BY A γ QUANTUM

We consider the process (1'). It is described in the lowest approximation of the weak and electromagnetic interactions by the sum of the two diagrams a and b of Fig. 1. The matrix element of such a process, based on the V-A coupling, has the form

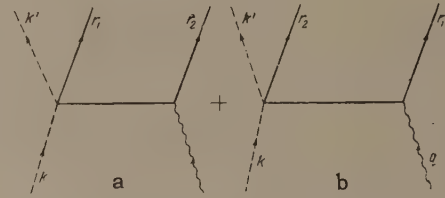


FIG. 1. Feynman diagram for process (1'); k and k' are the momenta of the incoming and scattered neutrino, q is the momentum of the photon, and r_1 and r_2 are the corresponding momenta of the positron and the electron.

$$M = \frac{Ge}{\sqrt{2}} \frac{1}{\sqrt{2\omega_q}} \bar{u}(r_2) \left[\hat{e} \frac{1}{\hat{r}_2 - \hat{q} - m} \gamma_\alpha (1 + \gamma_5) + \gamma_\alpha (1 + \gamma_5) \frac{1}{-\hat{r}_1 + \hat{q} - m} \hat{e} \right] u(r_1) (\bar{u}(k') \gamma_\alpha (1 + \gamma_5) u(k)), \quad (3)$$

where G is the weak-interaction constant, e is the electric charge, u are spinors, and $\hat{e} = e_\mu \gamma_\mu$.

The calculations lead to the following formula for the differential cross section

$$d\sigma_{ph} = \frac{4e^2 G^2}{(2\pi)^5 (kq)} \left\{ \frac{(r_1 q)(kq)(r_2 k')}{[(r_1 - q)^2 - m^2]^2} + \frac{(r_2 q)(k'q)(r_1 k)}{[(r_2 - q)^2 - m^2]^2} + [(r_1 - q)^2 - m^2]^{-1} [(r_2 - q)^2 - m^2]^{-1} [(2r_1 r_2 - r_1 q - r_2 q)(r_1 k)(r_2 k') - (r_1 r_2)(r_1 k)(qk') - (r_1 r_2)(r_2 k')(qk) + (qr_1)(kr_2)(k'r_2) + (qr_2)(r_1 k)(r_1 k')] \right\} \times \delta^4(k + q - k' - r_1 - r_2) \frac{d^3 r_1}{E_{r_1}} \frac{d^3 r_2}{E_{r_2}} \frac{d^3 k'}{E_{k'}}. \quad (4)$$

We have left out of this expression the terms proportional to m_e^2 (or m_μ^2). Inasmuch as the energy of the particles participating in the reaction is much greater than their masses, we shall henceforth neglect the particle mass where possible.

Integrating (4) we obtain the cross section for the process (1'):

$$\sigma_{ph} = \frac{2}{9\pi^2} \alpha G^2 \omega^2 \left(\ln \frac{\omega}{m} - \frac{19}{12} \right), \quad (5)$$

where ω is the total energy in the c.m.s. of the photon and the neutrino, m is the electron (muon), mass, and $\alpha = 1/137$.

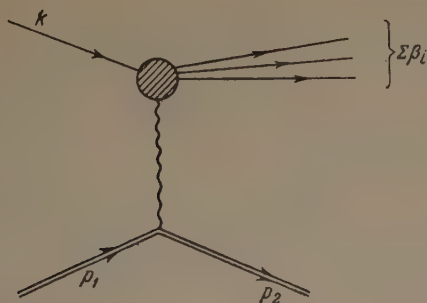


FIG. 2. Feynman diagram for the multiple production of particles in the Coulomb field of the nucleus. p_1 and p_2 are the initial and final momenta of the nucleus, k is the momentum of the incoming particle, and $\Sigma\beta_i$ is the totality of created particles with a summary momentum Q .

For the cross section of the process (2') we obtain the expression

$$\sigma_{ph} = \frac{1}{18\pi^2} \alpha G^2 w^2 \left(\ln \frac{w}{m} - \frac{55}{48} \right). \quad (6)$$

2. PROCEDURE FOR USING THE PHOTON CROSS SECTIONS FOR PROCESSES IN A COULOMB FIELD

An invariant formulation of the method was proposed by Gribov, Kolkunov, Okun', and Shekhter.^[4] The multiple production of particles in the Coulomb field of a nucleus is described in the lowest order by the diagram of Fig. 2. If we consider a nucleus with zero spin,* we obtain for the reaction cross section

$$\sigma = \frac{F^2(q)}{q^4} Z^2 e^2 \frac{\Phi_{\mu\nu}(p_1 + p_2)_\mu (p_1 + p_2)_\nu}{4 V(k p_1)^2 - k^2 p_1^2} \frac{d^3 p_2}{(2\pi)^3 E_2}. \quad (7)$$

Here $F(q)$ is the electromagnetic form of the nucleus, and $\Phi_{\mu\nu}$ is a tensor describing the upper vertex of the diagram (Fig. 2). From the gauge invariance requirements we obtain^[4]

$$\Phi_{\mu\nu} = a [(kq) \delta_{\mu\nu} + q^2 k_\mu k_\nu / (kq) - k_\mu q_\nu - k_\nu q_\mu] + b [q^2 \delta_{\mu\nu} - q_\mu q_\nu], \quad (8)$$

where a and b are scalar functions.

It can be readily verified that as $q^2 \rightarrow 0$ the quantity a goes into σ_{ph} , the cross section of the photoprocess $q + k \rightarrow \Sigma\beta_i$. Practically nothing can be said beforehand concerning the quantity β unless the tensor $\Phi_{\mu\nu}$ is calculated directly. We introduce the positive quantity $\Delta^2 = -q^2$ and make the following change of variables in (7)

$$\frac{d^3 p_2}{E_2} = \frac{\pi d\Delta^2 dw^2}{2M|k|},$$

where w is the energy of the aggregate of the particles $\Sigma\beta_i$ in their center-of-mass system. Substituting (8) in (7) we get

$$\sigma = \frac{Z^2 \alpha}{4\pi v} \frac{d\Delta^2 dw^2}{\Delta^4} F^2(\Delta) \left\{ a(w^2, \Delta^2) \left[\frac{2\Delta^2}{w^2 + \Delta^2 - m_l^2} + \frac{\Delta^2}{ME} - \frac{w^2 + \Delta^2 - m_l^2}{2E^2} \right] + b(w^2, \Delta^2) \frac{\Delta^2}{E^2} \left[1 + \frac{\Delta^2}{4M^2} \right] \right\}, \quad (9)$$

where M is the mass of the nucleus, E the energy of the incoming particle in the system where the nucleus is at rest, and v and m_l are the velocity and mass of the particle respectively.

Formula (9) is an exact expression for the cross section, but it is rather cumbersome to use for calculations. We shall show below that when the charged leptons in processes (1) and (2) are muons we can make several simplifications in (9). In this case the form of the expression for the cross section σ is found to be the same as in the formulas obtained by the Weizsäcker-Williams method. The conditions that lead to such a result are:

- 1) $a(w^2, \Delta^2)$ must differ from $a(w^2, 0)$ in the entire domain of variation of w^2 ;
- 2) the function $b(w^2, \Delta^2)$ must not increase more strongly than $a(w^2, \Delta^2)$ with increasing w^2 ;
- 3) all the terms in the curly brackets of (9) must be small compared with $a(w^2, \Delta^2) \cdot 2\Delta^2 / (w^2 + \Delta^2 - m_l^2)$.

The first condition is satisfied if $\Delta_{max}^2 \lesssim m^2$, where m is the mass of the charged lepton. The second condition is satisfied in our case, and this can be verified without calculating the tensor $\Phi_{\mu\nu}$. In order to explain how the third condition is satisfied in our case, let us examine the ranges of the variables Δ^2 and w^2 .

The range of variation of these variables, allowed by the conservation laws, is shown in Fig. 3. However, only part of this domain is essential to the process (shaded). The point is that the presence in (9) of the electromagnetic form factor of the nucleus $F(\Delta)$, which decreases sharply with increasing Δ , leads to an effective cut-off of the possible momentum transfer. For our relatively rough estimates we can approximate the form factor by

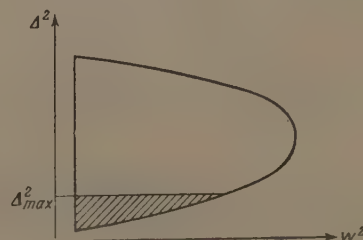


FIG. 3

*I. Ya. Pomeranchuk and I. M. Shumashkevich have shown that additional small terms appear in formula (7) in the case of nuclei with spin $1/2$.

$$F(\Delta) = \int \rho \exp(i\Delta r) d^3r / \int \rho d^3r,$$

where ρ is the distribution of the charge in the nucleus by means of the function

$$F(\Delta) = \begin{cases} 1, & \Delta \leq 1/R, \\ 0, & \Delta > 1/R \end{cases} \quad (10)$$

where R is the radius of the nucleus. Thus, we should take for the maximum value of the momentum transfer

$$\Delta_{\max}^2 = \frac{1}{R^2} = \mu^2 A^{-2/3}, \quad (11)$$

where μ is the mass of the pion and A the atomic weight of the nucleus.

We note that the condition 2) is therefore satisfied in the case of muons.

At high incident-particle energy the lower limit of the integration region is determined from the condition^[5]

$$\Delta_{\min}^2 \approx M^2 (\omega^2 - m_i^2)^2 / [(W^2 + M^2 - m_i^2)^2 - 4W^2 M^2], \quad (12)$$

where M is the mass of the nucleus and $W = \sqrt{(M+E)^2 - \mathbf{k}^2}$ is the energy in the c.m.s. of the incoming particle and of the original nucleus.

If we now integrate with respect to Δ^2 in formula (9) and neglect in the resultant expression terms of order $\mu/EA^{1/3}$ and $\mu/MA^{1/3}$, we obtain for the cross section

$$\sigma = \frac{Z^2 \alpha}{\pi} \int \sigma_{\text{ph}}(\omega^2) \frac{d\omega^2}{\omega^2} \ln \frac{2\mu EA^{-1/3}}{\omega^2} \eta, \quad (13)$$

which in fact is the Weizsäcker-Williams formula.

We wish to emphasize, however, that the situation in this case is the complete opposite of the electrodynamic case, for which the Weizsäcker-Williams method was developed. The point that is in electrodynamics $\sigma_{\text{ph}}(\omega^2)$ decreases with increasing ω^2 , and consequently small ω^2 and small Δ^2 are essential to the process. In our case the opposite holds true: $\sigma_{\text{ph}}(\omega^2)$ increases with increasing ω^2 , and consequently large momentum transfers are significant. Only the presence of the electromagnetic form factor of the nucleus leads to our result, and this furthermore is true only for muons. For electrons the substitution for $\sigma_{\text{ph.el}}(\omega^2)$ instead of a (ω^2, Δ^2) would be too crude an approximation.

We note finally that the formula for the process (1) does not reflect the fact that this is a threshold process. The reason is that the calculations were made for energies $E \gg m$, and we have neglected the lepton mass in the phase volume of the final state.

3. CROSS SECTIONS OF PROCESS (1) AND (2)

In the case when the energy of the incoming particle is much greater than the masses of the light particles participating in the process, the

factor η under the logarithm sign in formula (13) tends to unity. Then, substituting into (13) expression (5) for σ_{ph} (taking only the principal term in the formula for σ_{ph}), we obtain for the cross section of process (1)

$$\sigma = \frac{2Z^2 \alpha^2 G^2 \mu A^{-1/3}}{9\pi^3} E \left(\ln \frac{2\mu EA^{-1/3}}{m^2} - 2 \right). \quad (14)$$

We wish to make the following remark concerning this formula. If we integrate over the entire region of Fig. 3 without imposing any limitations on the maximum momentum transfer, then the cross section will contain terms proportional to $E^2 \ln(E/m)$ and E^2 . Such an energy dependence of the cross section, calculated without account of the form factor of the nucleus and consequently incorrect, was obtained, in particular, by Badalyan and Chou Kuang-chao^[6] for the process $\nu + A \rightarrow \nu + A + e^+ + e^-$.

It follows from the foregoing that the form factor of the nucleus exerts an appreciable influence on the energy dependence of the cross section.

The cross section of process (2) is obtained by substituting (6) into (13):

$$\sigma = \frac{Z^2 \alpha^2 G^2 \mu A^{-1/3}}{18\pi^3} E \left(\ln \frac{2\mu EA^{-1/3}}{m^2} - 2 \right). \quad (15)$$

The process (2) was considered for nonrelativistic electrons, without account of the form factor, by Gandel'man and Pinaev.^[7]

The numerical values of the cross section of the process (1) with an initial-neutrino energy $E = 10$ Bev are $\sigma \approx 6.6 \times 10^{-44} \text{ cm}^2$ and $\sigma \approx 6.0 \times 10^{-41} \text{ cm}^2$ for hydrogen and lead, respectively.

The authors are grateful to L. B. Okun' for suggesting the problem and for discussions, and thank I. Ya. Pomeranchuk for a discussion and for interest in the work.

¹ C. Weizsäcker, Z. Physik **88**, 612 (1934); E. Williams, Phys. Rev. **45**, 729 (1934).

² I. Ya. Pomeranchuk and I. M. Shmushkevich, Nucl. Phys. **23**, 452 (1961).

³ Wang, Fischer, Ciulli, and Ciulli, JETP **40**, 676 (1961), Soviet Phys. JETP **12**, 473 (1961).

⁴ Gribov, Kolkunov, Okun', and Shekhter, JETP, in press.

⁵ C. Chew and F. Low, Phys. Rev. **113**, 1640 (1959).

⁶ A. Badalyan and Chou Kuang-chao, JETP **38**, 664 (1960), Soviet Phys. JETP **11**, 477 (1960).

⁷ G. M. Gandel'man and V. S. Pinaev, JETP **37**, 1072 (1959), Soviet Phys. JETP **10**, 764 (1960).

THE COLLECTIVE GYROMAGNETIC RATIO FOR ODD ATOMIC NUCLEI

Yu. T. GRIN' and I. M. PAVLICHENKOV

Submitted to JETP editor April 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 954-958 (September, 1961)

A general expression for the gyromagnetic ratio of even-odd deformed nuclei is obtained by taking pair-correlation effects into account. It is demonstrated that the gyromagnetic ratio of odd proton nuclei is greater, and that of odd neutron nuclei smaller, than the ratio of neighboring even-even nuclei. The results are compared with the experiments.

THE gyromagnetic ratios of several nuclei in the rare-earth region have already been measured.^[1] The gyromagnetic ratios for excited rotational states of even-even nuclei are found to be of the order 0.2–0.4. For odd-proton nuclei, g_R^p is greater than the gyromagnetic ratio for the neighboring even-even nuclei, while for odd-neutron nuclei g_R^n is less.

Although the ratio Δ_p/Δ_n ^[2] ($\Delta_{n,p}$ is a quantity characterizing the pair correlation) can be calculated from the gyromagnetic ratio of an even-even nucleus, it is a difficult quantity to measure. It is therefore of interest to obtain a general expression for g_R of an odd nucleus and to relate it with the gyromagnetic ratio for the even nucleus.

The gyromagnetic ratio is defined by

$$g_R = \langle \mu \rangle / \langle I \rangle, \quad (1)$$

where $\langle \mu \rangle$ and $\langle I \rangle$ are the mean values of the magnetic moment and the angular momentum of the nucleus. One of the authors, together with Drozdov and Zaretskii,^[3] has calculated the density matrix of an odd rotating system. With the aid of this matrix it is easy to calculate the mean values of the angular momentum and magnetic moment operators

$$\mu = \sum (g_l I + g_s s).$$

The summation is over all the nucleons of the nucleus.

If the angular velocity Ω of the rotating system is parallel to the x axis, which in turn is perpendicular to the symmetry axis z of the nucleus, then $\langle I \rangle = J\Omega$, where J is the moment of inertia of the nucleus and

$$\langle \mu \rangle = [J_p + (g_s^p - 1) W^p + g_s^n W^n] \Omega. \quad (2)$$

The value of W is given by

$$W^{n,p} = \sum_{\lambda\lambda'} \frac{(E_\lambda E_{\lambda'} - \epsilon_\lambda \epsilon_{\lambda'} - \Delta_{n,p}^2) j_{\lambda\lambda'}^x s_{\lambda\lambda'}^x - i \Delta_{n,p} (\epsilon_\lambda - \epsilon_{\lambda'}) s_{\lambda\lambda'}^x f_{\lambda\lambda'}}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})} + \sum_{\lambda} \frac{2(E_{\lambda_0}^2 + \epsilon_\lambda \epsilon_{\lambda_0} + \Delta_{n,p}^2) j_{\lambda\lambda_0}^x s_{\lambda\lambda_0}^x + i \Delta_{n,p} (\epsilon_\lambda - \epsilon_{\lambda_0}) (s_{\lambda\lambda_0}^x f_{\lambda\lambda_0} - s_{\lambda_0\lambda}^x f_{\lambda\lambda_0})}{E_{\lambda_0} (\epsilon_\lambda^2 - \epsilon_{\lambda_0}^2)},$$

Here ϵ_λ is the single-particle energy reckoned from the Fermi surface ϵ_0 , and λ_0 is the state of the odd particle. The subscripts p and n denote summation in (3) over the proton and neutron states. The neutron and proton systems are assumed in this paper to be non-interacting.

The value of f is determined from the integral equation (see [4]). If we write $f = f^0 + f'$, where f^0 is the solution of the integral equation for the even-even nucleus and f' is a quasi-classical small change in the solution to account for the presence of the odd particle, then expression (3) can be rewritten

$$W^{n,p} = W_e^{n,p}(\kappa_0) - \sum_{\lambda\lambda'} \frac{i \Delta_{n,p} (\epsilon_\lambda - \epsilon_{\lambda'}) s_{\lambda\lambda'}^x f_{\lambda\lambda'}}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})} + \sum_{\lambda} \frac{2(E_{\lambda_0}^2 + \epsilon_\lambda \epsilon_{\lambda_0} + \Delta_{n,p}^2) j_{\lambda\lambda_0}^x s_{\lambda\lambda_0}^x + 2i \Delta_{n,p} (\epsilon_\lambda - \epsilon_{\lambda_0}) s_{\lambda\lambda_0}^x f_{\lambda\lambda_0}^0}{E_{\lambda_0} (\epsilon_\lambda^2 - \epsilon_{\lambda_0}^2)}, \quad (4)$$

$$W_e^{n,p}(\kappa_0) = \sum_{\lambda\lambda'} \frac{(E_\lambda E_{\lambda'} - \epsilon_\lambda \epsilon_{\lambda'} - \Delta_{n,p}^2) j_{\lambda\lambda'}^x s_{\lambda\lambda'}^x - i \Delta_{n,p} (\epsilon_\lambda - \epsilon_{\lambda'}) s_{\lambda\lambda'}^x f_{\lambda\lambda'}^0}{2E_\lambda E_{\lambda'} (E_\lambda + E_{\lambda'})}, \quad (5)$$

while κ_0 is determined from the values of Δ and β of the odd nucleus; in order of magnitude we have $\kappa \sim \epsilon_0 \beta / \Delta A^{1/3}$.

It is convenient to write the general expressions for the gyromagnetic ratios g_R^n and g_R^p of the odd-neutron and odd-proton nuclei in the form

$$g_R^n = g_R J_e / J_0 + g_s^n [W_e^n(\kappa_0) - W_e^n(\kappa_e)] / J_0 + \delta g_R^n, \quad (6)$$

$$g_R^p = 1 - (1 - g_R) J_e / J_0 + (g_s^p - 1) [W_e^p(\kappa_0) - W_e^p(\kappa_e)] / J_0 + \delta g_R^p;$$

$$g_R = [J_e^p + g_s^n W_e^n + (g_s^p - 1) W_e^p] / J_e. \quad (7)$$

$$E_\lambda = \sqrt{\epsilon_\lambda^2 + \Delta_{n,p}^2}. \quad (3)$$

g_R is the gyromagnetic ratio of the even nucleus. An analogous expression for g_R with account of f^0

was obtained by Nilsson and Prior.^[5] It is readily seen from (4) and (5) that

$$\delta g_R^{n,p} = \frac{g_s^{n,p} - g_l^{n,p}}{J_0} \left[\sum_{\lambda} \frac{2(E_{\lambda_0}^2 + \epsilon_{\lambda} \epsilon_{\lambda_0} + \Delta_{n,p}^2) j_{\lambda\lambda_0}^x s_{\lambda\lambda_0}^x + 2i\Delta_{n,p} (\epsilon_{\lambda} - \epsilon_{\lambda_0}) s_{\lambda\lambda_0}^x f_{\lambda_0\lambda}^0}{E_{\lambda_0} (\epsilon_{\lambda}^2 - \epsilon_{\lambda_0}^2)} - \sum_{\lambda\lambda'} \frac{2i\Delta_{n,p} (\epsilon_{\lambda} - \epsilon_{\lambda'}) s_{\lambda\lambda'}^x f_{\lambda'\lambda}^0}{2E_{\lambda} E_{\lambda'} (\epsilon_{\lambda} + \epsilon_{\lambda'})} \right]. \quad (8)$$

On going from the even nucleus to the odd one, the deformation remains practically unchanged, and the change in the pairing energy Δ is determined by the following expression^[2]
 $[\Delta = (\Delta_0 + \Delta_e)/2]$

$$(\Delta_e - \Delta_0)/\Delta = 1/2 \rho_0 \Delta \sim A^{-1/3}.$$

Since the function $W^{n,p}$ is smooth, its variation as N or Z changes by unity is

$$W(\kappa_e) - W(\kappa_0) \sim W(\kappa_e) \delta\kappa/\kappa \sim W(\kappa_e) A^{-1/3}.$$

Let us show further that $W \sim JA^{-2/3}$. For this purpose we consider the first term in (5) (without the small term containing f^0):

$$W_1 = \sum_{\lambda\lambda'} \left[1 - g \left(\frac{\epsilon_{\lambda} - \epsilon_{\lambda'}}{2\Delta} \right) \right] j_{\lambda\lambda'}^x s_{\lambda\lambda'}^x \delta(\epsilon_{\lambda}).$$

As shown by Migdal^[2] with a rectangular potential well model, it is necessary to retain in this sum only the values of λ' for which $m' = m \pm 1$ and $\nu = \nu'$, where m is the eigenvalue of the projection of the angular momentum on the symmetry axis and ν is the totality of the remaining quantum numbers. With quasi-classical accuracy the matrix element s^x for such transitions is equal to

$$s_{\lambda\lambda'}^x = \pm j_{\lambda\lambda'}/2l,$$

where the plus sign corresponds to $j = l + 1/2$ and the minus sign to $j = l - 1/2$ (l is the quantum number of the orbital angular momentum), and therefore W_1 will contain two sums over $j = l + 1/2$ and $j = l - 1/2$, with opposite signs; these sums cancel in the quasi-classical approximation. W_1 is equal to the next term in the expansion in $A^{-1/3}$, the order of which is $JA^{-2/3}$. Analogous arguments can be cited for the second term in W , since $f^0 \sim j^x$; as a result we get $W \sim JA^{-2/3}$. Naturally, the same estimate for W is obtained also in the second limiting model, an oscillator nuclear potential with spin-orbit coupling.

For heavy nuclei $g_s - g_l$ is numerically of the order of $A^{1/3}$, so that we obtain for the second term in (6) and (7) the estimate $(g_s - g_l) [W(\kappa_0) - W(\kappa_e)]/J \sim A^{-2/3}$.

Let us examine the expression for δg_R . Since $f' = f_0/2\rho_0\Delta \sim f_0 A^{-1/3}$, we obtain for the same reason as before

$$(g_s - g_l) \sum_{\lambda\lambda'} \frac{i\Delta (\epsilon_{\lambda} - \epsilon_{\lambda'}) s_{\lambda\lambda'}^x f_{\lambda'\lambda}^0}{2E_{\lambda} E_{\lambda'} (\epsilon_{\lambda} + \epsilon_{\lambda'})} \sim JA^{-2/3},$$

and since the first term in (8) contains a single sum, its order is $A^{-1/3}$. Terms containing f^0 in the expression for δg_R do not exceed 20 percent of the principal term and can therefore be estimated by using the expression derived by Migdal^[2] for f^0 in an oscillator nuclear potential.

Discarding terms of order $A^{-2/3}$ in (6) and (7) we obtain an approximate expression for the gyromagnetic ratio of odd nuclei when $\epsilon_{\lambda_0} = 0$:

$$g_R^n = g_R J_e/J_0 + \delta g_R^n, \quad (9)$$

$$g_R^p = 1 - (1 - g_R) J_e/J_0 + \delta g_R^p; \quad (10)$$

$$\delta g_R^{n,p} = \frac{(g_s - g_l)}{J_0 \Delta} \sum_{\lambda} s_{\lambda\lambda_0}^x j_{\lambda\lambda_0}^x \left(\frac{1}{\nu_{\lambda}^2} - \frac{g_1 + g_2}{g_1 \nu_1^2 + g_2 \nu_2^2} \right);$$

$$\nu_{\lambda} = \frac{(\epsilon_{\lambda} - \epsilon_{\lambda_0})}{2\Delta}, \quad \nu_1 = \frac{\omega_0}{2\Delta}, \quad \nu_2 = \frac{\omega_0}{\Delta},$$

$$g_x = \frac{\arg \operatorname{sh} x}{x \sqrt{1+x^2}}, \quad (11)^*$$

where $g_1 = g(\nu_1)$ and $g_2 = g(\nu_2)$.

In (9) and (10) δg_R is always smaller than the first term, by a factor $A^{-1/3}$. Consequently we can neglect in the approximate analysis its contribution to $g_R^{n,p}$. Recognizing that in the rare-earth region we have in the mean $J_e^n/J_0^n = 0.65$, $J_e^p/J_0^p = 0.9$ and $g_R = 0.3$, we get $g_R^n \approx 0.2$ and $g_R^p \approx 0.35$, i.e., the relation $g_R^p > g_R > g_R^n$ is satisfied in the mean.

In a more exact calculation of the gyromagnetic ratio of odd nuclei we must take δg_R into account. Although this term is smaller than the first term, it fluctuates more strongly, owing to the matrix element $s_{\lambda\lambda_0}^x$, and in some cases it may make an appreciable contribution to (9) and (10).

The gyromagnetic ratios $g_R^{n,p}$ were calculated from formulas (9), (10), and (11), and since the spin-orbit coupling is essential for δg_R , this quantity was calculated in the Nilsson model.^[6] The values of Δ_n and Δ_p were taken from the paper by Nilsson and Prior.^[5] The values of the gyromagnetic ratio g_R of an even-even nucleus having one less particle than the original odd nucleus were taken from the same paper,^[5] while

*sh = sinh.

Table I

Nucleus	State of odd particle $Nn_Z \Lambda \Omega$	J_e/J_o	g_R	δg_R	g_{R1}	g_R^p	
						Theory	Experiment
$^{153}_{90}\text{Eu}$	4 1 3 $5/2$	0.64	0.34	-0.14	0.58	0.44	0.45 ± 0.01
$^{159}_{94}\text{Tb}$	4 1 1 $3/2$	0.82	0.32	-0.07	0.44	0.37	0.24 ± 0.09
$^{165}_{98}\text{Ho}$	5 2 3 $7/2$	0.83	0.30	0.11	0.42	0.53	0.30 ± 0.07
$^{175}_{104}\text{Lu}$	4 0 4 $7/2$	0.99	0.31	-0.03	0.32	0.29	0.30 ± 0.05
$^{181}_{108}\text{Ta}$	4 0 4 $7/2$	0.97	0.28	-0.08	0.30	0.22	0.33 ± 0.01
$^{185}_{110}\text{Re}$	4 0 2 $5/2$	0.95	0.28	-0.02	0.31	0.29	0.41 ± 0.04
$^{187}_{112}\text{Re}$	4 0 2 $5/2$	0.93	0.35	-0.03	0.40	0.37	0.41 ± 0.04

Table II

Nucleus	State of odd particle $Nn_Z \Lambda \Omega$	J_e/J_o	g_R	δg_R	g_{R1}	g_R^n	
						Theory	Experiment
$^{155}_{91}\text{Gd}$	5 2 1 $3/2$	0.59	0.37	0.05	0.22	0.27	0.34 ± 0.07
$^{157}_{93}\text{Gd}$	5 2 1 $3/2$	0.74	0.33	0.04	0.24	0.28	0.22 ± 0.06
$^{161}_{95}\text{Dy}$	6 4 2 $5/2$	0.43	0.32	-0.05	0.14	0.09	0.25 ± 0.10
$^{167}_{99}\text{Er}$	6 3 3 $7/2$	0.65	0.30	-0.08	0.19	0.11	0.12 ± 0.05
$^{173}_{103}\text{Yb}$	5 1 2 $5/2$	0.87	0.31	0.01	0.27	0.28	0.20 ± 0.09
$^{177}_{105}\text{Hf}$	5 1 4 $7/2$	0.85	0.25	0.04	0.21	0.25	0.21 ± 0.01
$^{179}_{107}\text{Hf}$	6 2 4 $9/2$	0.71	0.25	-0.08	0.18	0.10	0.20 ± 0.03

the moments of inertia were obtained by experiment. It was found that δg_R amounts in the mean to 20 — 25 percent of the first term in (9) or (10). The values obtained for the gyromagnetic ratios of odd nuclei are listed in Tables I and II for protons and neutrons respectively. In these tables $g_{R1}^{n,p}$ corresponds to the first term in the expressions (9) or (10).

The theory is in qualitative agreement with experiment. No pretense of quantitative agreement can be made at present since the known experimental values of $g_R^{n,p}$ are subject to considerable errors. In addition, this quantity is not measured in the experiments directly, but is obtained indirectly from the probabilities of the magnetic transitions and the magnetic moments. On the other hand, it must be noted that the wave functions of Nilsson potential do not yield correct values of the magnetic moments of the odd nuclei. We are therefore not even assured of the correctness of the values of δg_R calculated in the Nilsson model.

This circumstance may change the theoretical value of g_R by 10 — 20 percent.

¹E. M. Bernstein and J. de Boer, Nucl. Phys. 18, 40 (1960).

²A. B. Migdal, JETP 37, 249 (1959), Soviet Phys. JETP 10, 176 (1960).

³Grin', Drozdov, and Zaretskii, JETP 38, 222 (1960), Soviet Phys. JETP 11, 162 (1960).

⁴Grin', Drozdov, and Zaretskii, JETP 38, 1297 (1960), Soviet Phys. JETP 11, 936 (1960).

⁵S. G. Nilsson and O. Prior, The Effect of Pair Correlation on the Moment of Inertia and the Collective Gyromagnetic Ratio of Deformed Nuclei, Preprint.

⁶S. G. Nilsson, Mat.-Fys. Medd. Dan. Vid. Sesk. 29, 1 (1955).

Translated by J. G. Adashko

THE NEUTRON STRENGTH FUNCTION ON THE OPTICAL MODEL

Yu. P. ELAGIN, V. A. LYUL'KA, and P. É. NEMIROVSKII

Submitted to JETP editor April 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 959-962 (September, 1961)

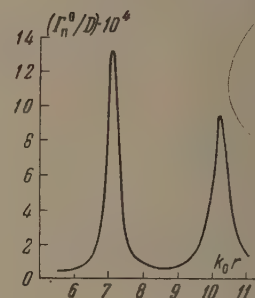
We treat the effect of surface absorption on the behavior of the strength function for s neutrons in the region $90 < A < 130$. We show that surface absorption gives no new results beyond those for volume absorption, and is unable to resolve the discrepancy between theory and experiment in this region. We also show that including the spin-orbit coupling in computations for deformed nuclei leads to an additional splitting of the maxima of the strength function and makes possible an improved agreement between theory and experiment both in the region of the maximum at $A \sim 150$ as well as the maximum at $A \sim 50$.

1. INTRODUCTION

CALCULATIONS of the strength function for $l = 0$ neutrons have been made many times. In most computations it is assumed that the absorption occurs throughout the volume of the nucleus and that the real and imaginary parts of the optical potential have the same r dependence. One then gets a correct description of the behavior of the strength function $f = \Gamma_n^0/D$ (where Γ_n^0 is the neutron width referred to an energy of 1 ev, and D is the level spacing) as a function of atomic weight. Computations with volume absorption for nonspherical nuclei have improved the agreement with experiment in the region of large deformations. But in the region of the minimum of the strength function at $90 < A < 130$, the experimental data are considerably below the theoretical values. Two proposals have been made for eliminating this difficulty:^[1] 1) introduction of surface absorption in place of volume absorption should improve the agreement with experiment; 2) assumption of an anomalously small value for the absorption in the region where N and Z are close to the magic number 50.

The first assumption was investigated in a paper of Khanna and Tang.^[2] Their results indicated a different behavior of the strength function for surface and for volume absorption. But this result contradicts other work^[3] in which it is asserted that the surface and volume absorptions differ very little at higher energies. We have therefore once again undertaken a computation of the strength function for surface absorption. Besides, the computations for deformed nuclei have been made previously without including the spin-orbit coupling. The present paper investigates for the first time the effect of spin-orbit interaction on the strength function for deformed nuclei.

FIG. 1. Neutron strength function with surface absorption, for $b = 0.5$ and $W_0 = 10$ Mev.



2. THE SPHERICAL NUCLEUS WITH SURFACE ABSORPTION

We have calculated the strength function for s -neutrons for the case of a spherical nucleus with surface absorption. The real part of the potential is the same as that used in computations with volume absorption, and as was done previously,^[4] we may write

$$V(r) = -V_0 [1 + \exp \{(r - R)/a\}]^{-1}, \quad (1)$$

where $V_0 = 50$ Mev, $a = 0.65$ fermis, $R = 1.24 A^{1/3}$ fermi. For the imaginary part of the potential we took the expression^[5]

$$W(r) = -W_0 \exp \{-(r - R)^2/b^2\}, \quad (2)$$

which corresponds to a surface absorption.

Computations were made for three cases: 1) $b = 0.5$, $W_0 = 10$ Mev; 2) $b = 1$, $W_0 = 5$ Mev and 3) $b = 1$, $W_0 = 10$ Mev.

For the first case ($b = 0.5$), the curve for the strength function f as a function of Ra^{-1} coincides for $W_0 = 2.5$ Mev^[4] to within 10% with the result for the volume absorption

$$W(r) = \zeta V(r), \quad (3)$$

where ζ is the ratio of the imaginary to the real part of the potential (cf. Fig. 1, in which the curves for volume and surface absorption coincided).

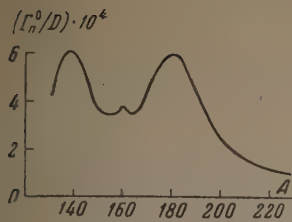


FIG. 2. Neutron strength function in the region $130 < A < 220$.

For the second and third cases ($b = 1$ and $W_0 = 5$ Mev or 10 Mev) the curves, which are not shown in Fig. 1, also were close to those for volume absorption.

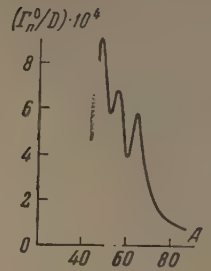
In contradiction to the assertion of Khanna and Tang,^[2] the use of the potential (2) gives no new results at all for the strength function and does not eliminate the difficulties cited in Sec. 1. The computation showed that $f_{\max}/f_{\min}$ is determined by the quantities W_0 and b , while the value of $(f_{\max}f_{\min})^{1/2}$, just as in the case of volume absorption, is determined primarily by the parameter a in formula (1), which affects the reflection at the nuclear surface.

As for the possibility of an anomalously small absorption near $N = 50$ and $Z = 50$, this assumption seems questionable since there are indications^[6] that for higher energies the absorption in this region of atomic weight is not significantly different from that in neighboring regions. Thus the question of the behavior of the strength function for $90 < A < 130$ stills remains open.

3. DEFORMED NUCLEI

We know that nuclei have a static deformation for $150 < A < 190$ and for $A > 220$. The axis of symmetry of the nucleus rotates relative to the space-fixed axes. Various authors^[7] have shown that the motion of the neutron-nucleus system can be described by a wave function which depends on the coordinates of the neutron in the laboratory system and on the Euler angles which characterize the position of the nuclear axis in space. The potential $V(r, \vartheta')$ representing the interaction of the neutron with the nucleus depends on the angle ϑ' between the direction of the nucleon and the axis of the nucleus. The quantity $V(r, \vartheta')$ can be expanded in a series of Legendre polynomials, and one obtains for the radial wave functions a system of equations which couple waves with different l' and l which satisfy the condition $l' + l = J$, where l' is the angular momentum of the nucleus and J is the total angular momentum of the system. If we keep only the terms in $P_2(\cos \vartheta')$ in the potential and set all higher terms equal to zero, then for deformations $\beta \leq 0.25$, we mix into the wave function with $l = 0$

FIG. 3. Strength function in the region $40 < A < 80$.



only the state with $l = 2$. (For zero energy, when there is no excitation of rotational levels, the state with $l = 2$ is different from zero only inside the nucleus.) An admixture of the state with $l = 4$ can occur only for very large deformations.

In theories without spin-orbit coupling, neutrons with $l = 2$ are described by a single wave function and, as a result, for slow neutrons one gets a system of two equations. For the strength function calculated in this way, the maximum at $A = 150$ splits into two maxima.

The importance of the spin-orbit interaction for neutrons with $l \neq 0$ has been shown in many papers. It is obvious that if we add to the potential $V(r, \vartheta')$ a term of the form $l \cdot \mathbf{s} r^{-1} \partial V(r, \vartheta') / \partial r$, the scattering of slow neutrons will now be described by a system of three rather than two equations (one for $l = 0$, and two for $l = 2$ corresponding to $j = 5/2$ and $j = 3/2$). Thus in this case the spin-orbit interaction should also have an effect on the interaction of slow neutrons with a deformed nucleus. The corresponding computations will be given in more detail in a paper concerning the scattering of neutrons of higher energy by deformed nuclei.

As the computation showed, the inclusion of spin-orbit interaction led to an additional splitting of the peaks in the strength function curve. Figure 2 shows the strength function computed for even-even nuclei on the following assumptions: $I' = 0.2$, $\beta = 0.15$, the energy of the rotational level $I = 2$, $E_2 = 90$ kev; we used a volume absorption $W = \zeta V$, with $\zeta = 0.05$. As one sees, in addition to the two maxima found in^[7] there is a bump on the curve in the region of the trough ($A \sim 160$) between these peaks, which corresponds to the situation observed experimentally. Thus the introduction of spin-orbit coupling improves the agreement between theory and experiment in the region of the giant resonance $A \sim 160$.

The giant resonance at $A \sim 50$ also exhibits a complex structure (as indicated by experiment). In particular it appears from the experimental data that one can conclude that there are bumps on the curve at $A \approx 65$ and $A \approx 75$. In this re-

gion of atomic weights ($40 < A < 80$) the nucleus again does not have a static spherical shape. But here the deformations are not static, but rather dynamic, resulting from asymmetric vibrations of the nuclear surface. Thus the conditions for applicability of the adiabatic theory, with a rotating deformed nucleus, are not fulfilled. However one may attempt to apply it in this region also (without any pretense at good quantitative agreement with experiment). Results of computations including spin-orbit interaction and using the parameters cited above are given in Fig. 3. Apparently the curve gives a qualitative picture of the bumps on the sides of the giant resonance.

In any case it appears obvious to us that including the nuclear deformation enables one to explain the relatively large values of the strength function in the region of $A = 65 - 75$, which are not gotten from the simple model of a spherical well.

It is a pleasure to express our gratitude to Z. D. Dobrokhotova for interest and help in programming of the numerical computations.

¹A. M. Lane et al., Phys. Rev. Letters **2**, 424 (1959).

²F. C. Khanna and Y. C. Tang, Nuclear Phys. **15**, 337 (1960).

³H. Amster, Phys. Rev. **113**, 911 (1959).

⁴P. É. Nemirovskii, Sovremennye Modeli Atomno-yadra (Present-day Models of the Atomic Nucleus) Atomizdat, 1960.

⁵F. E. Bjorklund and S. Fernbach, Phys. Rev. **109**, 1295 (1958).

⁶P. E. Nemirovskii, JETP **36**, 588 (1959), Soviet Phys. JETP **9**, 408 (1959).

⁷B. Margolis and E. S. Troubetzkoy, Phys. Rev. **106**, 105 (1957); V. V. Vladimirovsky and I. L. Ilyina, Nuclear Phys. **6**, 295 (1958); Chase, Wilets, and Edmonds, Phys. Rev. **110**, 1080 (1958).

Translated by M. Hamermesh
165

PRODUCTION OF HIGH-ENERGY π MESON BEAMS

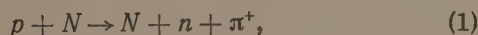
Yu. P. NIKITIN, I. Ya. POMERANCHUK, and I. M. SHMUSHKEVICH

Submitted to JETP editor April 25, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 963-968 (September, 1961)

The cross section for the production of fast π^+ mesons in collisions between high-energy protons and nuclei in which a small momentum is transferred to the nucleus is calculated. Under these conditions, the main role in the interaction is played by the exchange of virtual photons. The cross section under consideration was therefore determined by means of available experimental data on the photoproduction cross section of π^+ mesons on protons.

FAST π meson beams are at present produced in proton accelerators as the result of an interaction of primary protons with nuclei. If these protons have an energy E_L of the order of 30–40 BeV then, as will be shown below, a considerable role is played by the Coulomb field of the nucleus in the production of π^+ mesons with energy $\epsilon_L \approx 0.4 - 0.5 E_L$. The reaction involved is



in which a small momentum is transferred to the nucleus N . If the condition

$$q^2 \ll \mu^2 A^{-2/3}, \quad (2)$$

is satisfied, where $q^2 = (P - P')^2$ are the 4-momenta of the nucleus before and after collision, A is the atomic weight of the nucleus, and μ the π meson mass, then, as has been shown earlier,^[1] the main contribution to process (1) is due to the diagrams involving the exchange of virtual photons (see Fig. 1). Under these conditions, the differential cross section of process (1) $d\sigma$ may be determined by the Weizsäcker-Williams method, using the available experimental data on the photoproduction cross section of π mesons on the nucleus. According to the relation between these cross sections [see ^[1], Eq. (2.17)], we have

$$d\sigma = \frac{Z^2 \alpha}{\pi} \frac{d\omega^2}{\omega^2 - m^2} \frac{dq^2}{(q^2)^2} \left[q^2 - \left(\frac{\omega^2 - m^2}{2E_L} \right)^2 \right] d\sigma_p \quad (\alpha = 1/137) \quad (3)$$

where Z is the nuclear charge, E_L is the energy of the incident proton in the laboratory system (l.s.), m is the proton mass, ω is the energy of the neutron and the π^+ meson produced in the reaction (or of the proton and the π^0 meson) in the c.m.s., and $d\sigma_p$ is the differential cross section for the photoproduction

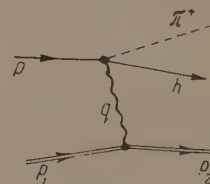


FIG. 1

which we shall write in the following way:

$$d\sigma_p = 2\pi\sigma_p(\omega, \theta_c) \sin\theta_c d\theta_c, \quad (5)$$

where θ_c is the angle between the momenta of the π meson and of the γ ray in the c.m.s., and ω is the frequency in the W coordinate system (the system in which the proton is at rest before the collision).

Taking into account that $w^2 = m^2 + 2m\omega$, we integrate Eq. (3) over q^2 in the limits between

$$q_{min}^2 = \left(\frac{\omega^2 - m^2}{2E_L} \right)^2 = \left(\frac{m\omega}{E_L} \right)^2 \text{ and } q_{max}^2 = \frac{\mu^2}{A^{2/3}}.$$

Denoting the result obtained by $\sigma_\pi(E_L, \omega, \theta_c) d\omega \sin\theta_c d\theta_c$, we have

$$\sigma_\pi(E_L, \omega, \theta_c) d\omega \sin\theta_c d\theta_c = 2Z^2 \alpha \left[2 \ln \frac{\mu E_L}{m A^{1/3} \omega} - 1 + \left(\frac{m A^{1/3} \omega}{\mu E_L} \right)^2 \right] \frac{d\omega}{\omega} \sigma_p(\omega, \theta_c) \sin\theta_c d\theta_c. \quad (6)$$

It should be noted that, in view of condition (2) and since $w^2 - m^2 < 2E_L \sqrt{q^2}$, the permissible values of ω are determined by the condition

$$\omega < \mu E_L / m A^{1/3}. \quad (7)$$

In order to obtain the energy and angular distributions of π mesons in the l.s., we shall pass in Eq. (6) from θ_c and ω to the variables ϵ_L and θ_L (denoting the energy and the angle of emission of the produced π meson in the l.s. measured from the direction of the primary-proton momentum).

From the conservation laws for the process (4), it follows that

$$\cos \theta = [(m + \omega) \varepsilon - m\omega - \frac{1}{2}\mu^2] / g\omega, \quad (8)$$

where ε and g are the energy and momentum of the produced π meson in the W system, and θ is the angle between the momenta of the π meson and of the γ ray in the same system. On the other hand

$$\varepsilon_L = m^{-1}(\varepsilon E_L - g p_L \cos \theta). \quad (9)$$

From Eqs. (8) and (9), we have

$$\varepsilon = \frac{\omega(1 - \varepsilon_L/p_L) + \mu^2/2m}{1 - (\omega/m)(E_L/p_L - 1)}. \quad (10)$$

Taking Eq. (7) into account, we can neglect the second term in the denominator of the expression for ε . In fact,

$$\frac{\omega}{m} \left(\frac{E_L}{p_L} - 1 \right) \approx \frac{\omega}{m} \frac{m^2}{2E_L^2} < \frac{m}{2E_L^2} \frac{\mu E_L}{mA^{1/2}} = \frac{\mu}{2E_L A^{1/2}} \ll 1. \quad (11)$$

In addition, in the numerator of Eq. (10), we can neglect the term $\mu^2/2m$, since, at any rate, $\varepsilon > \mu$. Consequently, with sufficiently good accuracy, we have

$$\varepsilon = \omega(1 - \varepsilon_L/E_L). \quad (12)$$

We shall now consider those π mesons which are emitted in the l.s. at small angles to the direction of primary protons, and which have a very high energy ε_L constituting a large fraction of the energy E_L . As we shall soon see, these π mesons will be relativistic in the W system, i.e., for them $\varepsilon \approx g \gtrsim \mu$. Taking this into account, we obtain from (9) and (12)

$$\omega = m\varepsilon_L / (E_L - \varepsilon_L) [1 - (p_L/E_L) \cos \theta]. \quad (13)$$

Hence, for a given ε_L , we have for the minimum frequency $\omega_{\min}$ (corresponding to $\theta = \pi$)

$$\omega_{\min} = m\varepsilon_L / 2(E_L - \varepsilon_L). \quad (14)$$

The minimum value of ε corresponding to $\omega_{\min}$ is, according to Eq. (12), found to equal $m\varepsilon_L / 2E_L$. Therefore, the assumption made earlier that $\varepsilon \gtrsim \mu$ is fulfilled for those π mesons whose energy ε_L in the l.s. satisfies the condition

$$\varepsilon_L \gtrsim 2\mu E_L / m = \varepsilon_1. \quad (15)$$

On the other hand, for the maximum frequency $\omega_{\max}$ we obtain, from Eq. (13), the value of ω [$\omega_{\max} = 2\varepsilon_L E_L^2 / m(E_L - \varepsilon_L)$] which, for the large ε_L of interest, is always much greater than $\omega_{\max}$ determined by Eq. (7). Therefore, the integration with respect to ω in Eq. (6) should be carried out over the limits from $\omega_{\min}$ to $\omega_{\max} = \mu E_L / mA^{1/3}$.

Moreover, the condition $\omega_{\min} < \omega_{\max}$ determines the upper limit for ε_L which can be attained if the condition (7) is satisfied:

$$(\varepsilon_L)_{\max} = \frac{E_L}{1 + m/2\omega_{\max}} = \frac{E_L}{1 + m^2 A^{1/2} / 2\mu E_L} = \varepsilon_2. \quad (16)$$

Thus, the production of fast π mesons with energy $\varepsilon_L \gtrsim 2\mu E_L / m$ in the Coulomb field of the nucleus is only possible for a sufficiently large energy E_L of the incident protons

$$E_L \gtrsim mA^{1/2} / (1 + 2\mu/m). \quad (17)$$

Let us denote the velocity of the c.m.s. with respect to the W system by V and the velocity of the produced π mesons in the c.m.s. by v :

$$V = \frac{\omega}{m + \omega}, \quad v = \sqrt{(\omega - \mu^2/2m)^2 - \mu^2} / (\omega + \mu^2/2m). \quad (18)$$

Then

$$v \cos \theta_c = \left[g \cos \theta - \frac{\omega}{m + \omega} \varepsilon \right] / \left[\varepsilon - \frac{\omega}{m + \omega} g \cos \theta \right]. \quad (19)$$

With the same accuracy with which we obtained Eq. (13), we have

$$\begin{aligned} \cos \theta_c &\approx \left[\cos \theta - \frac{\omega}{m + \omega} \right] / \left[1 - \frac{\omega}{m + \omega} \cos \theta \right] \\ &= 1 - \frac{\varepsilon_L}{E_L} \left(2 + \frac{m}{\omega} \right). \end{aligned} \quad (20)$$

Furthermore,

$$E_L \varepsilon_L - \sqrt{(E_L^2 - m^2)(\varepsilon_L^2 - \mu^2)} \cos \theta_L = m\varepsilon. \quad (21)$$

Hence, taking Eqs. (12) and (15) into account, we obtain

$$\begin{aligned} \cos \theta_L &= \frac{\varepsilon_L E_L}{\sqrt{(E_L^2 - m^2)(\varepsilon_L^2 - \mu^2)}} \left(1 - \frac{m\varepsilon}{\varepsilon_L E_L} \right) \\ &\approx \left(1 + \frac{m^2}{2E_L^2} + \frac{\mu^2}{2\varepsilon_L^2} \right) \left[1 - \frac{m\omega}{\varepsilon_L E_L} \left(1 - \frac{\varepsilon_L}{E_L} \right) \right] \\ &\approx 1 + \frac{m^2}{2E_L^2} - \frac{m\omega}{\varepsilon_L E_L} \left(1 - \frac{\varepsilon_L}{E_L} \right). \end{aligned} \quad (22)$$

Consequently, for small θ_L ($\theta_L \ll 1$), which is the only range important for the process under consideration, we have

$$\theta_L^2 = \frac{2m\omega}{\varepsilon_L E_L} \left(1 - \frac{\varepsilon_L}{E_L} \right) - \frac{m^2}{E_L^2}. \quad (23)$$

The maximum angle $\theta_L^{\max}(E_L, \varepsilon_L)$ of emission of a pion with energy ε_L permitted by Eq. (7) is therefore

$$[\theta_L^{\max}(E_L, \varepsilon_L)]^2 = \frac{2\mu}{A^{1/2} \varepsilon_L} \left(1 - \frac{\varepsilon_L}{E_L} \right) - \frac{m^2}{E_L^2}. \quad (24)$$

Equations (20) and (23) enable us to pass to variables ε_L and θ_L in Eq. (6), as a result of which we find

Table I. Cross section $\sigma_\pi(E_L, \epsilon_L) \times 10^{-30}$ cm²/Bev

ϵ_L / E_L	0,30	0,35	0,40	0,50
E_L , Bev				
20	131	59	15	0,02
30	194	108	36	1,9
40	226	129	50	5,3

$$\sigma_\pi(E_L, \epsilon_L, \theta_L) d\epsilon_L d\Omega_L = \frac{4Z^2\alpha}{\pi} \left[2 \ln \frac{\omega_{\max}}{\omega} - 1 + \left(\frac{\omega}{\omega_{\max}} \right)^2 \right] \sigma_p(\omega, \theta_c) \frac{\theta_L^2 + m^2/\epsilon_L E_L}{E_L(\theta_L^2 + m^2/E_L^2)} d\epsilon_L d\Omega_L. \quad (25)$$

In the above expression, ω and θ_c are determined for given ϵ_L and θ_L by Eqs. (20) and (23), and $d\Omega_L = 2\pi\theta_L d\theta_L$.

In order to obtain the energy distribution of the π mesons produced, it is more convenient to change in Eq. (6) directly from the variable θ_c to ϵ_L using Eq. (20), and to integrate the expression obtained with respect to ω between the limits given by Eqs. (7) and (14). We then obtain

$$\begin{aligned} \sigma_\pi(E_L, \epsilon_L) d\epsilon_L &= d\epsilon_L \int_{\omega_{\min}}^{\omega_{\max}} \sigma_\pi(\omega, \theta_c) \sin \theta_c \frac{d\theta_c}{d\epsilon_L} d\omega \\ &= 2Z^2 \frac{d\epsilon_L}{E_L} \alpha \int_{\omega_{\min}}^{\omega_{\max}} \left[2 \ln \frac{\omega_{\max}}{\omega} - 1 + \left(\frac{\omega}{\omega_{\max}} \right)^2 \right] \sigma_p(\omega, \theta_c) \left(2 + \frac{m}{\omega} \right) \frac{d\omega}{\omega}. \end{aligned} \quad (26)$$

It is convenient to represent the available data on the cross section for photoproduction of π mesons on nucleons in the form

$$\sigma_p(\omega, \theta_c) = A(\omega) + B(\omega) \cos \theta_c + C(\omega) \cos^2 \theta_c, \quad (27)$$

where $A(\omega)$, $B(\omega)$, and $C(\omega)$ are values known from experiments (see^[2], where data on the photoproduction of π^+ mesons for γ ray energies less than 1 Bev are collected). Taking (20) into account, this enables us to carry out a numerical integration in Eq. (26) and to obtain the energy spectrum of π mesons with energy ϵ_L in the range from $2\mu E_L/m$ to $E_L/(1 + m^2 A^{1/3}/2\mu E_L)$ [see Eqs. (15) and (16)] and emitted in a narrow cone with aperture $\theta_L^{\max}(E_L, \epsilon_L)$ [see Eqs. (24)].

The results for the cross section $\sigma_\pi(E_L, \epsilon_L)$ calculated in this manner for nuclei with $Z = 82$ and $A = 207$ (Pb) and primary-proton energies $E_L = 20, 30$, and 40 Bev for different values of ϵ_L are shown in Table I.

In connection with these calculations, it should be mentioned that the frequency ω corresponding

to the $\frac{3}{2}, \frac{3}{2}$ resonance is of the order of 2μ . Therefore, if $\omega_{\min} \gtrsim 2\mu$, the resonance range falls within the integration limits in (26). According to Eq. (14), this will take place if

$$\epsilon_L/E_L \lesssim (1 + m/4\mu)^{-1} \approx 0,4. \quad (28)$$

The main contribution to the integral for $\sigma_\pi(E_L, \epsilon_L)$ will then be due to frequencies ω close to $\omega_{\min}$. Therefore, if, in addition to (28), the following condition is satisfied

$$\omega_{\max}/\omega_{\min} \gg 1, \quad (29)$$

then, for the important values of ω , the first factor in the integrand of Eq. (26) also will be large (the logarithmic term being large). If conditions (28) and (29) are satisfied, we can assume that the mechanism due to the action of the Coulomb field of the nucleus will contribute greatly, and may constitute the main contribution, to the production of fast π mesons (i.e., may be greater than the contribution of purely nuclear interactions).

For π mesons with energy ϵ_L satisfying condition (28), the inequality (29) becomes

$$E_L/2mA^{1/3} \gg 1. \quad (30)$$

It is essential to satisfy (29) also because of the following reasons: passing from Eq. (3) to Eq. (6), we have used the exact equality, which determines the upper limit of integration with respect to q^2 , instead of the inequality (2) which gives the condition of applicability of Eq. (3). The inaccuracy due to this procedure will be negligible if the argument of the logarithm in the brackets of Eq. (6) $\omega_{\max}/\omega$ will be much greater than unity. Therefore, the corresponding error will also be small for $\sigma_\pi(E_L, \epsilon_L)$ if Eq. (29) is satisfied.

For a primary proton energy E_L equal, for example, to 30 Bev, and an energy of the produced π mesons $\epsilon_L = 0,4 E_L = 12$ Bev, condition (28) is satisfied and $\omega_{\max}/\omega_{\min} \approx 2,5$. We therefore see that, under these conditions, the mechanism discussed contributes greatly to the production of π mesons of the corresponding energies emitted at angles not greater than about 2° [see Eq. (24)]. Clearly, this mechanism will remain important in the range of π meson energies $\epsilon_L \gtrsim 0,4 E_L$, which will form an increasingly large energy fraction with increasing E_L .

The first row of Table II shows the values of the cross section $\sigma'_\pi(E_L)$ which were obtained by a numerical integration of $\sigma_\pi(E_L, \epsilon_L)$ within the limits given by Eqs. (15) and (16):

$$\sigma'_\pi(E_L) = \int_{\epsilon_1}^{\epsilon_2} \sigma_\pi(E_L, \epsilon_L) d\epsilon_L. \quad (31)$$

Table II. Cross sections $\sigma'_\pi(E_L)$ and $\sigma_\pi(E_L)$, 10^{-27} cm²

E_L , Bev	20	30	40	50	60	75
σ'_π	0.146	0.395	0.656			
σ_π	1.09	2.54	3.88	5.05	6.08	7.38

The second row of the table shows the data which indicate the dependence of the cross section $\sigma_\pi(E_L)$ on E_L corresponding to the production in the reaction (1) of π mesons with arbitrary momenta satisfying condition (2). This cross section corresponds to π mesons which, in the l.s., can also have small energy ($\epsilon_L < 2\mu E_L/m$). The magnitude of this cross section is obtained by introducing the total cross section $\sigma_p(\omega)$ for the reaction (4) into Eq. (6), instead of the differential cross section $2\pi\sigma_p(\omega, \theta_c) \sin \theta_c d\theta_c$, and then carrying out a numerical integration with respect to ω between the limits $\omega_1 = \mu(1 + \mu/2m)$ [the energy threshold of reaction (4)] and $\omega_{\max}$ [determined by Eq. (7)]

$$\sigma_\pi(E_L) = \frac{Z^2\alpha}{\pi} \int_{\omega_1}^{\omega_{\max}} \left[2 \ln \frac{\omega_{\max}}{\omega} - 1 + \left(\frac{\omega}{\omega_{\max}} \right)^2 \right] \sigma_p(\omega) \frac{d\omega}{\omega}. \quad (32)$$

The cross sections $\sigma_\pi(E_L, \epsilon_L)$ and $\sigma'_\pi(E_L)$ can be calculated using the available experimental data only for a primary proton energy $E_L < 40$ Bev. According to Eq. (7), $\omega_{\max}$ which, for $A = 207$, is greater than 1 Bev, corresponds to the energy E_L greater than 40 Bev. There are so far no corresponding data for the photoproduction cross section $\sigma_p(\omega, \theta_c)$ for a γ ray energy greater than 1 Bev. The cross section $\sigma_\pi(E_L)$ may, with good accuracy, also be calculated for $E_L > 40$ Bev. This is because the main contribution to the integral in the right-hand side of (32) is due to relatively low frequencies corresponding to the $3/2, 3/2$ resonance.

In a recently published article,^[3] Drell came to the conclusion that accelerated electrons rather than protons should result in a greater yield of ultra-high energy π mesons even at γ ray (produced in the bremsstrahlung of electrons) and proton energies $E_L = 25$ Bev. However, in esti-

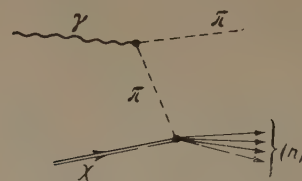


FIG. 2

imating the cross section for the production of fast π mesons by collision of protons with nucleons, Drell based himself on the results of the statistical model. The applicability of this model for processes in which all the energy of the colliding particles is transmitted to a single particle, is clearly very doubtful. On the other hand, the photoproduction of π mesons on nuclei was calculated by Drell using the polar approximation, i.e., taking into account the diagrams of the type shown in Fig. 2 [on which X and (n) correspond to the nucleus in the initial and final states]. In application to processes involving nuclei, the validity of such an approximation is especially doubtful (see the remarks of Gell-Mann^[4] referring to the article of Drell), since it is possible that the nearest singularity in the value of the momentum transferred to the nucleus in the amplitude of such processes is not due to the mass μ of the π mesons but to a smaller quantity $1/R = \mu/A^{1/3}$ (where R is the nuclear radius).

¹I. Ya. Pomeranchuk and I. M. Shmushkevich, Nucl. Phys. (in press).

²H. A. Bethe and F. de Hoffmann, Mesons and Fields (Row, Peterson, and Co., New York, 1955) vol. II; H. H. Bingham and A. B. Clegg, Phys. Rev. 112, 2053 (1958); F. P. Dixon and R. L. Walker, Phys. Rev. Lett. 1, 459 (1958).

³S. D. Drell, Phys. Rev. Lett. 5, 278 (1960).

⁴M. Gell-Mann, Proc. of the 1960 Ann. Int. Conf. on High-Energy Physics, Univ. of Rochester, 1960, p. 641.

THE THEORY OF ELECTROMAGNETIC FLUCTUATIONS IN A PLASMA

V. P. SILIN

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 26, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 969-976 (September, 1961)

We consider electromagnetic fluctuations in a non-equilibrium plasma both when there are no strong fields present and in the case when the plasma is in a strong magnetic field.

1. One can assume that at the present time the theory of thermal fluctuations in a plasma in thermodynamic equilibrium has already been developed.* The difference—which is essentially not one of principle—between the theory of electromagnetic fluctuations of a plasma and the one given, for instance, in the book by Landau and Lifshitz^[2] consists in the use of the complex dielectric-constant tensor with account of spatial dispersion. Such a theory of fluctuations can, however, not be used for an application to a rarefied plasma in which the collisions are extremely rare and which can thus stay in a thermodynamic non-equilibrium state for a long time. The present paper is devoted to a consideration of fluctuations in such a non-equilibrium plasma.† We then obtain expressions for the fluctuations by straightforward calculation, by regarding the plasma as a system of weakly interacting particles.

2. We introduce the Fourier components of the electrical field operator

$$\hat{E}(\omega, \mathbf{k}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \hat{E}(\mathbf{r}, t) e^{i\omega t - i\mathbf{k}\mathbf{r}}. \quad (1)$$

To find expressions for the fluctuations we must know the quantum-mechanical average of the following operator

$$\frac{1}{2} [\hat{E}_j(\omega, \mathbf{k}) \hat{E}_i(\omega', \mathbf{k}') + \hat{E}_i(\omega', \mathbf{k}') \hat{E}_j(\omega, \mathbf{k})]. \quad (2)$$

It is clear that the average value of such an operator can at once be written down if we know the explicit form of the electrical field operators. In the case of interest to us, of weakly interacting particles, one can easily obtain the explicit form of the field operators. To do this one must take into

account the fact that the form of the matrix elements of the field operators is determined by the matrix elements of the particle current density when the particles make a transition from one state to another.^[3]

We first consider a case when there is no strong field in the plasma. The states of the particles in such a plasma can be described by plane waves and the transition matrix element of the current density of particles of the kind *a*, which corresponds to a transition from a state *n* to a state *m*, is of the form

$$e_a \alpha_{mn} \exp \{i\hbar^{-1}[(\mathbf{p}_n - \mathbf{p}_m, \mathbf{r}) - (E_n - E_m)t]\}, \quad (3)$$

where α are the Dirac matrices. We can then write down for the Fourier component of the electrical field operator (1) the equation

$$\{(\omega/c)^2 \epsilon_{ij}(\omega, \mathbf{k}) - k^2 \delta_{ij} + k_i k_j\} \langle m | \hat{E}_j(\omega, \mathbf{k}) | n \rangle = -\frac{4\pi i \omega}{c} e_a \alpha_{mn}^i \delta\left(\hbar \mathbf{k} - \mathbf{p}_n + \mathbf{p}_m\right) \delta\left(\omega - \frac{E_n - E_m}{\hbar}\right), \quad (4)$$

where ϵ_{ij} is the dielectric-constant tensor.

We can then immediately write down an expression for the matrix element of the field corresponding to a transition of a particle from a state *n* to a state *m*:

$$\langle m | \hat{E}^i(\omega, \mathbf{k}) | n \rangle = -4\pi i \omega c^{-1} e_a A_{ij}^{-1}(\omega, \mathbf{k}) \alpha_{mn}^j \times \delta\left(\mathbf{k} - \frac{\mathbf{p}_n - \mathbf{p}_m}{\hbar}\right) \delta\left(\omega - \frac{E_n - E_m}{\hbar}\right), \quad (5)$$

$$A_{ij}(\omega, \mathbf{k}) = (\omega/c)^2 \epsilon_{ij}(\omega, \mathbf{k}) - k^2 \delta_{ij} + k_i k_j. \quad (6)$$

Equation (5) enables us easily to evaluate the expectation value of the operator (2). Using the fact that the number of particles of kind *a* per unit volume is equal to N_a and that the probability that the *a*-th particle has a momentum $\mathbf{p}_a$ [and accordingly an energy $E(\mathbf{p}_a)$] is determined by the distribution functions $f_a(\mathbf{p}_a)$, we have

$$\frac{1}{2} [\hat{E}_j(\omega, \mathbf{k}) \hat{E}_i(\omega', \mathbf{k}') + \hat{E}_i(\omega', \mathbf{k}') \hat{E}_j(\omega, \mathbf{k})] = \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') (E_i E_j)_{\omega, \mathbf{k}}, \quad (7)$$

*A survey of such a theory and the corresponding bibliography are given in the book by Rukhadze and the present author.^[1]

†The theory of non-equilibrium electrical fluctuations was considered by L. V. Keldysh—to whom the author is grateful for information—in connection with another problem.

where the bar indicates a quantum-mechanical average, and

$$(E_i E_j)_{\omega, \mathbf{k}} = \frac{1}{4} \sum_a \left(\frac{4\pi e_a \omega}{c} \right)^2 A_{il}^{-1}(\omega, \mathbf{k}) A_{jr}^{*-1}(\omega, \mathbf{k}) N_a \times \int d\mathbf{p}_a f_a(\mathbf{p}_a) \left\{ \delta \left[\omega - \frac{1}{\hbar} \{E(\mathbf{p}_a + \hbar \mathbf{k}) - E(\mathbf{p}_a)\} \right] \times \left(-\delta_{lr} \frac{[E(\mathbf{p}_a + \hbar \mathbf{k}) - E(\mathbf{p}_a)]^2 - c^2 \hbar^2 k^2}{2E(\mathbf{p}_a)E(\mathbf{p}_a + \hbar \mathbf{k})} + \frac{c^2 p_a^r (\mathbf{p}_a + \hbar \mathbf{k})^l}{E(\mathbf{p}_a)E(\mathbf{p}_a + \hbar \mathbf{k})} + \frac{c^2 p_a^l (\mathbf{p}_a + \hbar \mathbf{k})^r}{E(\mathbf{p}_a)E(\mathbf{p}_a + \hbar \mathbf{k})} \right) + \delta \left[\omega + \frac{1}{\hbar} \{E(\mathbf{p}_a - \hbar \mathbf{k}) - E(\mathbf{p}_a)\} \right] \times \left(-\delta_{lr} \frac{[E(\mathbf{p}_a - \hbar \mathbf{k}) - E(\mathbf{p}_a)]^2 - c^2 \hbar^2 k^2}{2E(\mathbf{p}_a)E(\mathbf{p}_a - \hbar \mathbf{k})} + \frac{c^2 p_a^r (\mathbf{p}_a - \hbar \mathbf{k})^l}{E(\mathbf{p}_a)E(\mathbf{p}_a - \hbar \mathbf{k})} + \frac{c^2 p_a^l (\mathbf{p}_a - \hbar \mathbf{k})^r}{E(\mathbf{p}_a)E(\mathbf{p}_a - \hbar \mathbf{k})} \right) \right\}. \quad (8)$$

The spectral functions for the vectors of the magnetic and the electrical induction $\mathbf{B}$ and $\mathbf{D}$ are connected with $(E_i E_k)_{\omega, \mathbf{k}}$ by the relation

$$(B_i B_j)_{\omega, \mathbf{k}} = c^2 \omega^{-2} k_r k_s e_{ir} e_{js} (E_i E_t)_{\omega, \mathbf{k}}; \\ (D_i D_j)_{\omega, \mathbf{k}} = e_{ir}(\omega, \mathbf{k}) e_{jl}^*(\omega, \mathbf{k}) (E_r E_l)_{\omega, \mathbf{k}}$$

In the classical limit $\hbar \rightarrow 0$

$$(E_i E_j)_{\omega, \mathbf{k}} = \sum_a \left(\frac{4\pi e_a \omega}{c} \right)^2 A_{il}^{-1}(\omega, \mathbf{k}) A_{jr}^{*-1}(\omega, \mathbf{k}) \times N_a \int d\mathbf{p}_a f_a(\mathbf{p}_a) \frac{v_a^l v_a^r}{c^2} \delta(\omega - \mathbf{k} \mathbf{v}_a), \quad (9)$$

where $\mathbf{v}_a$ is the velocity of the a -th particle.

We point out that the spectral formula for the fluctuations of the Lorentz force acting upon particle a is equal to*

$$(e_a^2 (\mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}])_i (\mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}])_j)_{\omega, \mathbf{k}} = \sum_b \left(\frac{4\pi e_a e_b \omega}{c^2} \right)^2 \times \int d\mathbf{p}_b N_b f_b \delta(\omega - \mathbf{k} \mathbf{v}_b) v_b^l v_b^r \left(\delta_{is} \left[1 - \frac{\mathbf{k} \mathbf{v}_a}{\omega} \right] + \frac{k_i v_a^s}{\omega} \right) A_{sl}^{-1}(\omega, \mathbf{k}) A_{tr}^{*-1}(\omega, \mathbf{k}) \left(\delta_{jt} \left[1 - \frac{\mathbf{k} \mathbf{v}_a}{\omega} \right] + \frac{k_j v_a^t}{\omega} \right).$$

This last formula can be used to write down the diffusion coefficient in the transport equation for charged particles.

In the particular case of particles distributed isotropically over the plasma, when the dielectric-constant tensor is of the form

$$\epsilon_{ij}(\omega, \mathbf{k}) = (\delta_{ij} - k_i k_j / k^2) \epsilon^t(\omega, k) + k_i k_j k^{-2} \epsilon^l(\omega, k), \quad (10)$$

and therefore

$$A_{ij}^{-1}(\omega, \mathbf{k}) = \frac{\delta_{ij} - k_i k_j / k^2}{(\omega/c)^2 \epsilon^t(\omega, k)} + \frac{c^2}{\omega^2 \epsilon^l} \frac{k_i k_j}{k^2}, \quad (11)$$

Equation (9) becomes of the form

$$(E_i E_j)_{\omega, \mathbf{k}} = \sum_a \left(\frac{4\pi e_a}{k^2} \right)^2 \int d\mathbf{p}_a N_a f(\mathbf{p}_a) \delta(\omega - \mathbf{k} \mathbf{v}_a) \times \left\{ \frac{k_i k_j}{|\epsilon^l(\omega, k)|^2} + \frac{1}{2} \frac{\omega^2 [\mathbf{k} \mathbf{v}_a]^2 (k^2 \delta_{ij} - k_i k_j)}{|\omega^2 \epsilon^t(\omega, k) - c^2 k^2|^2} \right\}. \quad (12)$$

In the case of an equilibrium particle distribution, Eq. (12) and, of course, also Eqs. (8) and (9), go over into the formulae following from the theory of thermal fluctuations.^[1]

3. We turn now to the case of a plasma in a strong magnetic field $\mathbf{B}_0$. We note first that it is clear from what we have already said that the fluctuations of the electrical field can be written in the form*

$$\frac{1}{2} [\hat{E}_l(\omega, \mathbf{k}) \hat{E}_l(\omega', \mathbf{k}') + \hat{E}_l(\omega', \mathbf{k}') \hat{E}_l(\omega, \mathbf{k})] = - (4\pi/c^2) \omega \omega' \times A_{il}^{-1}(\omega', \mathbf{k}') A_{jr}^{-1}(\omega, \mathbf{k}) I_{rl}(\omega, \mathbf{k}; \omega', \mathbf{k}'). \quad (13)$$

On the right hand side stands the correlation function of the transition current

$$I_{rl}(\omega, \mathbf{k}; \omega', \mathbf{k}') = \frac{1}{2} [\hat{j}_r(\omega, \mathbf{k}) \hat{j}_l(\omega', \mathbf{k}') + \hat{j}_l(\omega', \mathbf{k}') \hat{j}_r(\omega, \mathbf{k})],$$

and we now turn to evaluate this. We note in passing that one should not confuse the transition-current correlation function with the induced-current correlation function, which can easily be found by using the field equations provided, naturally, that the electrical field correlation function, for instance, is known.

We restrict ourselves to a non-relativistic analysis. Moreover, we shall not take spin effects into account. For the evaluation of the matrix elements we use the Landau representation, and we use thus for the complete set the value p_n^z of the component of the momentum parallel to $\mathbf{B}_0$, the energy pertaining to the transverse motion $E^\perp(l_n) = \hbar \Omega_a (l_n + 1/2)$ (here $\Omega_a = |e_a| B_0 / \mu_a c$), and finally $y_{0n} = -cp_n^x / e_a B_0$, the projection of the center of the Larmor orbit upon the y axis. We have then for the matrix element of the Fourier component of the transition current^[4,5]

$$\langle m | \hat{j}_a(\omega, \mathbf{k}) | n \rangle = \delta(\omega - [l_n - l_m] \Omega_a - p_n^z k_z / \mu_a + \hbar k_z^2 / 2 \mu_a) \times \delta \left(k_z - \frac{p_n^z - p_m^z}{\hbar} \right) \delta \left(k_x + \frac{e_a B_0}{\hbar c} [y_{0n} - y_{0m}] \right) \times \exp \left\{ -\frac{1}{2} i k_y (y_{0m} + y_{0n}) \right\} \langle l_m | \hat{j}_a(\mathbf{k}) | l_n \rangle, \quad (14)$$

*We note that without taking further conditions, say boundary conditions, into account, Eq. (13) and also Eqs. (5), (8), (9), and (12) can become meaningless in the region where the plasma is transparent, i.e., under conditions where the determinant of the tensor $A_{ij}(\omega, \mathbf{k})$ tends to zero. This latter fact corresponds to the circumstance that in the transparency region the non-equilibrium fluctuations of the electromagnetic field are not determined by the states of the charges. This observation arose from a conversation with M. A. Leontovich to whom I should like to express my gratitude.

* $[\mathbf{v}_a \mathbf{B}] = \mathbf{v}_a \times \mathbf{B}$.

where

$$\begin{aligned}\langle l_m | \hat{j}_a^x(k) | l_n \rangle &= |e_a| \Omega_a i \frac{\partial}{\partial k_y} \langle l_m | e^{-ik_y y} | l_n \rangle, \\ \langle l_m | \hat{j}_a^y(k) | l_n \rangle &= i \hbar k_x \frac{|e_a|}{2\mu_a} \langle l_m | e^{-ik_y y} | l_n \rangle + i e_a \sqrt{\hbar \Omega_a / 2\mu_a} \\ &\times \{ \sqrt{l_m} \langle l_m - 1 | e^{-ik_y y} | l_n \rangle - \sqrt{l_n} \langle l_m | e^{-ik_y y} | l_n - 1 \rangle \}, \\ (15)\end{aligned}$$

$$\begin{aligned}\langle l_m | \hat{j}_a^z(k) | l_n \rangle &= \frac{e_a}{\mu_a} \left(p_n^z - \frac{1}{2} \hbar k_z \right) \langle l_m | e^{-ik_y y} | l_n \rangle, \\ \langle l_m | e^{-ik_y y} | l_n \rangle &= l! (l_n! l_m!)^{-1/2} e^{-X/2} L_l^{l_m-l_n}(X) \\ &\times \left\{ \sqrt{\frac{c\hbar}{z|e_a|B_0}} [k_x \text{sign } e_a (l_n - l_m) - ik_y] \right\}^{l_m-l_n}. \quad (16)\end{aligned}$$

In the last equation $l = \min(l_m, l_n)$, $X = c\hbar k_\perp^2 / 2|e_a|B_0$, $k_\perp^2 = k_x^2 + k_y^2$, and

$$L_s^r(X) = \sum_{t=0}^{\infty} \binom{n+r}{n-t} \frac{1}{t!} (-X)^t$$

is a Laguerre polynomial.

Equations (14) to (16) enable us to determine the transition-current correlation function. We get thus, for the case of a spatially uniform distribution for which the density matrix $f_a(p_z, n)$ is independent of y_0

$$\begin{aligned}I_{ij}(\omega, k; \omega', k') &= \delta(\omega + \omega') \delta(k + k') \int dp_z \sum_{n=0}^{\infty} N_a f_a(p_z, n) 2\pi \frac{\hbar |e_a| B_0}{c} \\ &\times \sum_{m=0}^{\infty} \left\{ \delta \left(\omega - [m - n] \Omega_a - \frac{p_n^z k_z}{\mu_a} - \frac{\hbar k_z^2}{2\mu_a} \right) \langle n | \hat{j}_a^i(k) | m \rangle \langle m | \hat{j}_a^j(-k) | n \rangle \right. \\ &+ \delta \left(\omega + [m - n] \Omega_a - \frac{p_n^z k_z}{\mu_a} + \frac{\hbar k_z^2}{2\mu_a} \right) \langle n | \hat{j}_a^j(-k) | m \rangle \langle m | \hat{j}_a^i(k) | n \rangle \left. \right\}. \quad (17)\end{aligned}$$

This equation determines the fluctuations of the random transition current and, combined with Eq. (13), also the electromagnetic field fluctuations.

The fluctuation formulae can appreciably be simplified in the classical limit $\hbar = 0$. We can in that case use the asymptotic representation of the Laguerre polynomials and we have

$$\begin{aligned}\langle l_m | \hat{j}_a^z(k) | l_n \rangle &= e_a v_z (-1)^{l_m-l_n} J_{|l_m-l_n|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \\ &\times [i(k_y/k_\perp) + (k_x/k_\perp) \text{sign } e_a (l_m - l_n)]^{l_m-l_n}, \quad (18)\end{aligned}$$

$$\begin{aligned}\langle l_m | \hat{j}_a^x(k) | l_n \rangle &= |e_a| \Omega_a (-1)^{l_m-l_n} i \frac{\partial}{\partial k_y} \left\{ J_{|l_m-l_n|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \right. \\ &\times [i(k_y/k_\perp) + (k_x/k_\perp) \text{sign } e_a (l_m - l_n)]^{l_m-l_n} \left. \right\}, \quad (19)\end{aligned}$$

$$\begin{aligned}\langle l_m | \hat{j}_a^y(k) | l_n \rangle &= \frac{i}{2} e_a v_\perp (-1)^{l_m-l_n} \left\{ i \frac{k_y}{k_\perp} \right. \\ &+ \frac{k_x}{k_\perp} \text{sign } e_a (l_m - l_n + 1) \left. \right\}^{l_m-l_n+1} \\ &\times J_{|l_m-l_n+1|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) - J_{|l_m-l_n-1|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \\ &\times \left[i \frac{k_y}{k_\perp} + \frac{k_x}{k_\perp} \text{sign } e_a (l_m - l_n - 1) \right]^{l_m-l_n-1} \left. \right\}, \quad (20)\end{aligned}$$

where $v_z = p_n^z / \mu_a$, $v_\perp = (2E_n / \mu_a)^{1/2}$, $J_l(z)$ — a Bessel function.

Bearing in mind the fact that the matrix elements (18) and (19) do not change when we interchange the states m and n , while the matrix element (20) changes sign, we can use these equations to write down an equation which determines the fluctuations of the random currents:

$$\begin{aligned}I_{ij}(\omega, k; \omega', k') &= \delta(\omega + \omega') \delta(k_z + k'_z) \delta(k_x + k'_x) \sum_a e_a^2 N_a \\ &\times \int_{-\infty}^{+\infty} dp_z \int_0^\infty p_\perp dp_\perp \int_{-\infty}^{+\infty} dy_0 f_a(p_z, p_\perp, y_0) \sum_{l=-\infty}^{+\infty} \delta(\omega + l\Omega_a \\ &- k_z v_z) \exp \{ -i(k_y + k'_y) y_0 \} F_i^{(1)}(k_x, k_y) F_j^{(2)}(k_x, k'_y), \quad (21)\end{aligned}$$

$$\begin{aligned}F_x^{(1)}(k_x, k_y) &= F_x^{(2)}(k_x, k_y) = v_\perp (-1)^l \left[i \frac{k_y}{k_\perp} \right. \\ &+ \frac{k_x}{k_\perp} \text{sign}(l e_a) \left. \right]^{l-1} \left\{ i \frac{k_y}{k_\perp} J_{|l|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \text{sign } e_a \right. \\ &- \frac{k_x}{k_\perp} \frac{\Omega_a}{k_\perp v_\perp} l J_{|l|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \left. \right\}, \quad (22)\end{aligned}$$

$$\begin{aligned}F_y^{(1)}(k_x, k_y) &= F_y^{(2)}(k_x, k_y) (-1) \\ &= i v_\perp (-1)^l \frac{1}{2} \left\{ J_{|l+1|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \left[i \frac{k_y}{k_\perp} \right. \right. \\ &+ \frac{k_x}{k_\perp} \text{sign}(e_a [l+1]) \left. \right]^{l+1} \\ &- \left[i \frac{k_y}{k_\perp} + \frac{k_x}{k_\perp} \text{sign}(e_a [l-1]) \right]^{l-1} J_{|l-1|} \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \left. \right\}, \quad (23)\end{aligned}$$

$$\begin{aligned}F_z^{(1)}(k_x, k_y) &= F_z^{(2)}(k_x, k_y) = v_z (-1)^l J_{|l|} (k_\perp v_\perp / \Omega_a) \\ &\times \left[i \frac{k_y}{k_\perp} + \frac{k_x}{k_\perp} \text{sign}(e_a l) \right]^{l-1}. \quad (24)\end{aligned}$$

In the particular case of distributions which are independent of y_0 , the projection of the center of the Larmor orbit on the y axis, Eq. (21) becomes of the form ($k_y = 0$)

$$\begin{aligned}I_{ij}(\omega, k; \omega', k') &= \delta(\omega + \omega') \delta(k + k') \sum_a N_a e_a^2 \\ &\times 2\pi \int_{-\infty}^{+\infty} dp_z \int_0^\infty p_\perp dp_\perp f_a(p_z, p_\perp) \sum_{l=-\infty}^{+\infty} \delta(\omega - l\Omega_a - k_z v_z) \pi_{ij}, \quad (25)\end{aligned}$$

where the tensor π_{ij} has the following components

$$\begin{aligned}
\pi_{xx} &= v_{\perp}^2 [l(\Omega_a/k_{\perp}v_{\perp})J_l(k_{\perp}v_{\perp}/\Omega_a)]^2, \\
\pi_{yy} &= v_{\perp}^2 [J_l'(k_{\perp}v_{\perp}/\Omega_a)]^2, \quad \pi_{zz} = v_z^2 J_l^2(k_{\perp}v_{\perp}/\Omega_a), \\
\pi_{xy} &= -\pi_{yx} \\
&= iv_{\perp}^2 l J_l(k_{\perp}v_{\perp}/\Omega_a) J_l'(k_{\perp}v_{\perp}/\Omega_a) (\Omega_a/k_{\perp}v_{\perp}) \operatorname{sign} e_a, \\
\pi_{xz} &= \pi_{zx} = v_{\perp} v_z l J_l^2(k_{\perp}v_{\perp}/\Omega_a) (\Omega_a/k_{\perp}v_{\perp}) \operatorname{sign} k_x, \\
\pi_{zy} &= -\pi_{yz} = iv_{\perp} v_z J_l(k_{\perp}v_{\perp}/\Omega_a) J_l'(k_{\perp}v_{\perp}/\Omega_a) \operatorname{sign}(e_a k_x).
\end{aligned} \tag{26}$$

This tensor is exactly the same as the one arising in the theory of the dielectric susceptibility of a plasma in a constant magnetic field.^[1]

Equations (25) and (26) combined with (13) enable us to determine the classical fluctuations of the electromagnetic field in a plasma in a strong magnetic field. Stepanov and Kitsenko^[6] have given general formulae for the dielectric-constant tensor of a plasma with particle-distribution functions of the form $f(p_z, p_{\perp})$. As an application of our equations we obtain an expression for the fluctuations of the Coulomb field. We can then take

$$A_{il}^{-1}(\omega, \mathbf{k}) = \frac{k_l k_i}{k^2} \frac{c^2}{\omega^2} \frac{k^2}{k_s k_r e_{sr}(\omega, \mathbf{k})}. \tag{27}$$

In that case*

$$\begin{aligned}
(E_i E_j)_{\omega, \mathbf{k}} &= \sum_a \frac{(4\pi e_a)^2 N_a k_i k_j}{|k_s k_r e_{sr}(\omega, \mathbf{k})|^2} \\
&\times 2\pi \int_{-\infty}^{+\infty} dp_z \int_0^{\infty} dp_{\perp} p_{\perp} f_a(p_z, p_{\perp}) \sum_{l=-\infty}^{+\infty} \delta(\omega - l\Omega_a \\
&- k_z v_z) J_l^2\left(\frac{k_{\perp} v_{\perp}}{\Omega_a}\right).
\end{aligned} \tag{28}$$

We point out at this juncture that an expression similar to the one occurring on the right-hand side of Eq. (28) occurred earlier when the collision integral for charged particles in a strong magnetic field was written down.^[7,4] It is thus possible to express such a collision integral directly in terms of the fluctuations of the electromagnetic field.

We note in conclusion that the equations obtained here can also be used in the case where the particle-distribution function in the plasma depends on the coordinates and the time. A correct description of the fluctuations will then, clearly, be given for frequencies and wave numbers larger than the characteristic frequencies and wave numbers which determine the change in the non-equilibrium distribution function.

*The sum over l in Eq. (28) can also be written in the form of the following integral:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx J_0\left(2 \frac{k_{\perp} v_{\perp}}{\Omega_a} \sin \frac{\Omega_a x}{2}\right) \cos\{x(\omega - k_z v_z)\}.$$

APPENDIX

The author has shown^[8] that when there are no strong fields in the plasma the collision integral for charged particles is of the form

$$J_a = \frac{\partial}{\partial p_a^i} \left(D_{ij} \frac{\partial f_a}{\partial p_a^j} \right) - \frac{\partial}{\partial p_a^i} (A_i f_a), \tag{A.1}$$

where

$$\begin{aligned}
D_{ij} &= \sum_b N_b \int d\mathbf{p}_b f_b \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\frac{4\pi e_a e_b}{c^2} \right)^2 \pi k_i k_j \delta(\mathbf{k} \mathbf{v}_a \\
&- \mathbf{k} \mathbf{v}_b) |v_a^r A_{rl}^{-1}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) v_b^l|^2, \\
A_i &= \sum_b N_b \int d\mathbf{p}_b \frac{\partial f_b}{\partial p_b^i} \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\frac{4\pi e_a e_b}{c^2} \right)^2 \pi k_i k_j \delta(\mathbf{k} \mathbf{v}_a \\
&- \mathbf{k} \mathbf{v}_b) |v_a^r A_{rl}^{-1}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) v_b^l|^2.
\end{aligned}$$

It is of interest to express the diffusion coefficients D_{ij} and the friction coefficient A_i of Eq. (A.1) in terms of quantities which determine the electromagnetic properties of the plasma. To do this we note that

$$\begin{aligned}
\left(e_a^2 \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}] \right) \right)_i \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}] \right)_j \Big|_{\mathbf{k} \mathbf{v}_a, \mathbf{k}} &= \sum_b \left(\frac{4\pi e_a e_b}{c^2} \right)^2 \\
&\times N_b \int d\mathbf{p}_b f_b \delta(\mathbf{k} \mathbf{v}_a - \mathbf{k} \mathbf{v}_b) k_i k_j |v_a^r A_{rl}^{-1}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) v_b^l|^2.
\end{aligned} \tag{A.2}$$

Using this relation we get

$$D_{ij} = \int \frac{d\mathbf{k}}{(2\pi)^3} \pi \left(e_a^2 \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}] \right) \right)_i \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}] \right)_j \Big|_{\mathbf{k} \mathbf{v}_a, \mathbf{k}} \tag{A.2}$$

We note further that when there are no strong fields the following relation holds^[1] for the complex dielectric constant

$$\begin{aligned}
\varepsilon_{ij}(\omega, \mathbf{k}) - \varepsilon_{ji}^*(\omega, \mathbf{k}) \\
= -2\pi i \sum_a \frac{4\pi e_a^2}{\omega^2} N_a \int d\mathbf{p}_a \left(\mathbf{k} \frac{\partial f_a}{\partial \mathbf{p}_a} \right) v_a^i v_a^j \delta(\omega - \mathbf{k} \mathbf{v}_a),
\end{aligned}$$

which enables us to write the friction coefficient in the form

$$\begin{aligned}
A_i &= \frac{i}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} k_i \{ e_{rl}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) - \varepsilon_{lr}^*(\mathbf{k} \mathbf{v}_a, \mathbf{k}) \} \\
&\times 4\pi e_a^2 \left(\frac{\mathbf{k} \mathbf{v}_a}{c^2} \right)^2 v_a^l A_{rl}^{-1}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) v_a^s A_{st}^{*-1}(\mathbf{k} \mathbf{v}_a, \mathbf{k}).
\end{aligned} \tag{A.3}$$

We denote by $\mathbf{E}^{(a)}(\omega, \mathbf{k})$ the Fourier component of the electrical field produced in the plasma by particle a . It is clear that in our case such a component differs from zero only for $\omega = \mathbf{k} \cdot \mathbf{v}_a$ and is of the form

$$E_i^{(a)}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) = -(4\pi i e_a / c^2) \mathbf{k} \mathbf{v}_a A_{ij}^{-1}(\mathbf{k} \mathbf{v}_a, \mathbf{k}) v_j^a.$$

Taking this formula into account we can write Eq. (A.3) in the form

$$A_i = \frac{i}{2} \int \frac{dk}{(2\pi)^3} \frac{k_i}{4\pi} \{e_{rt}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) - e_{tr}^*(\mathbf{k}\mathbf{v}_a, \mathbf{k})\} E_r^{(a)}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) E_i^{(a)*}(\mathbf{k}\mathbf{v}_a, \mathbf{k}). \quad (\text{A.4})$$

Bearing in mind that the energy dissipated per unit time by the field $\mathbf{E}^{(a)}$ is equal to^[1]

$$\frac{i}{2} \int \frac{dk}{(2\pi)^3} \frac{\mathbf{k}\mathbf{v}_a}{4\pi} \{e_{rt}^*(\mathbf{k}\mathbf{v}_a, \mathbf{k}) - e_{tr}(\mathbf{k}\mathbf{v}_a, \mathbf{k})\} E_r^{(a)}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) E_i^{(a)*}(\mathbf{k}\mathbf{v}_a, \mathbf{k}),$$

we can say that Eq. (A.4) determines the force braking the particle a or, which comes to the same, the change in its momentum per unit time when we take the origin of the electromagnetic field accompanying the motion of the particle a into account.

¹A. A. Rukhadze and V. P. Silin, *Élektromagnitnye svoïstva plazmy i plazmopodobnykh sred* (Electromagnetic Properties of a Plasma and of Plasma-like Media), Atomizdat, 1951.

²L. D. Landau and E. M. Lifshitz, *Élektrodinamika sploshnykh sred* (Electrodynamics of Continuous Media), Gostekhizdat, 1957.

³C. Möller, *Z. Physik* **70**, 786 (1951).

⁴V. M. Eleonskii, P. S. Zyryanov, and V. P. Silin, *Fizika metallov i metallovedenie* (Physics of Metals and Metallurgy) **11**, No. 6 (1961).

⁵P. S. Zyryanov and V. P. Kalashnikov, *JETP* **41**, 1119 (1961), *Soviet Phys. JETP* **14**, in press.

⁶K. N. Stepanov and A. B. Kitsenko, *J. Tech. Phys. (U.S.S.R.)* **31**, 167 (1960), *Soviet Phys. Tech. Phys.* **6**, 120 (1961).

⁷N. Rostoker, *Phys. Fluids* **3**, 922 (1960).

⁸V. P. Silin, *JETP* **40**, 1768 (1961), *Soviet Phys. JETP* **13**, 1244 (1961).

Translated by D. ter Haar
167

CONTRIBUTION TO THE THEORY OF RELATIVISTIC COULOMB SCATTERING I.

V. G. GORSHKOV

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 28, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 977-984 (September, 1961)

The cross section for scattering of charged spinor particles in the Coulomb field of a nucleus is calculated by taking into account the finite size of the nucleus and screening. The relative accuracy of the calculation is of the order of $(\alpha Z)^2$. The calculations are all based on the Furry-Sommerfeld-Maue function and lead to an analytic formula for the cross section.

1. INTRODUCTION

THE cross section of elastic scattering of a relativistic spinor particle in a Coulomb field can be expressed in the form of a series in the partial moments.^[1] This series is not summed in the general case, and leads to numerical results with considerable difficulty. If only terms of order $(\alpha Z)^3$ are retained, this series can be summed,^[2] the sum being an exceedingly simple analytic function. It proved possible to obtain the same formula in the Born approximation.^[3]

Attempts to include terms of order $(\alpha Z)^4$ were made by Mitter and Urban,^[4] by perturbation theory, but the infrared divergences were incorrectly eliminated. The results obtained in that paper are presented in the form of cumbersome double integrals. In the nonrelativistic case, a proof of the elimination of the infrared divergences in terms of order $(\alpha Z)^4$ was given by Kacser.^[5]

In the present paper we develop a method in which terms of order $(\alpha Z)^4$ can be accounted for by using the Furry-Sommerfeld-Maue function. The radiation corrections, which are significant only at very small values of Z ($Z \sim 1$), are neglected. In part I of the paper, the finite nuclear dimensions and screening are taken into account in first approximation. We consider the conditions under which this is sufficient for the employed accuracy in $(\alpha Z)^4$. In part II of the article the finite nuclear dimensions and the screening will also be accounted for in the second approximation, and the results of numerical calculations will be given for certain values of the parameters. The final expression for the cross section can be represented in both cases in analytic form.

In a recent paper by Johnson, Weber, and Mullin^[6] analogous calculations are made for a pure

Coulomb potential using a somewhat different method. In the region covered by both investigations, the results are in agreement apart from algebraic transformations.

2. GENERAL EXPRESSION FOR THE SCATTERING AMPLITUDE

The scattering amplitude of a Dirac particle in an external statistical field V has the following form (see [7])

$$F(k, p) = 2\pi^2 \bar{u}_k \langle k | \hat{V} | \psi_p \rangle u_p = \bar{u}_k f(k, p) u_p, \quad (1)$$

where $\hat{V} = \gamma_4 V$, $k^2 = p^2 = E^2 - m^2$, $\hbar = c = 1$; and p and q are the momenta of the incoming and outgoing particles. The function $|k\rangle u_k$ is a solution of the free Dirac equation, while the function $|\psi_p\rangle u_p$ is the solution of the Dirac equation in a field V asymptotically represented by a plane and diverging wave

$$|\psi_p\rangle = |p\rangle + G\hat{V}|\psi_p\rangle = R|p\rangle,$$

$$\langle f | G | s \rangle = \frac{m - i\hat{f}}{f^2 - p^2 - i\varepsilon} \delta(f - s), \quad (2)$$

$$R = (1 - G\hat{V})^{-1} = \sum_{n=0}^{\infty} (G\hat{V})^n;$$

$$\hat{V}R = R^+\hat{V} \quad (f_0 = E; \quad \hat{f} = \gamma_i f_i). \quad (2a)$$

Assume that the field V can be represented as a sum of a strong field V_1 and a weak field V_2 . We introduce the solution of the Dirac equation in the field V_1 :

$$|\Phi_p\rangle = |p\rangle + G\hat{V}_1|\Phi_p\rangle = R_1|p\rangle, \quad (3)$$

where

$$R_1 = 1 + G\hat{V}_1 R_1 = (1 - G\hat{V}_1)^{-1}, \quad \hat{V}_1 R_1 = R_1^+ \hat{V}_1. \quad (3a)$$

It is then easy to verify that the following equation holds:

$$R = R_1 + R_1 \hat{G} \hat{V}_2 R = R_1 + R \hat{G} \hat{V}_2 R_1. \quad (4)$$

Taking (2a), (3a), and (4) into account we can obtain the following expression for the amplitude (1):

$$f(\mathbf{k}, \mathbf{p}) = f_1(\mathbf{k}, \mathbf{p}) + f_2(\mathbf{k}, \mathbf{p}), \quad (5)$$

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2 \langle \mathbf{k} | \hat{V}_1 | \Phi_p \rangle, \quad (5a)$$

$$f_2(\mathbf{k}, \mathbf{p}) = 2\pi^2 \{ \langle \Phi_k | \hat{V}_2 | \Phi_p \rangle + \langle \Phi_k | \hat{V}_2 R_1 \hat{G} \hat{V}_2 | \Phi_p \rangle + \langle \mathbf{k} | \hat{V}_2 R \hat{G} \hat{V}_2 R_1 \hat{G} \hat{V}_2 | \Phi_p \rangle \}, \quad (5b)$$

where $\langle \psi_{\mathbf{k}} | = \langle \mathbf{k} | R_1^\dagger$ corresponds to a function asymptotically represented by a plane and converging wave.

As shown earlier,^[8] the function $|\Phi_p\rangle$ can be represented in the first potential by

$$|\Phi_p\rangle = |\Phi_p^0\rangle + |\Phi_p\rangle, \quad (6)$$

with

$$|\Phi_p^0\rangle = |\mathbf{p}\rangle + G_0 V_1 |\Phi_p^0\rangle,$$

$$\langle \mathbf{f} | G_0 | \mathbf{s} \rangle = \frac{2E}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \delta(\mathbf{f} - \mathbf{s}), \quad (6a)$$

$$|\Phi_p\rangle = |\Phi_p^1\rangle + G \hat{V}_1 |\Phi_p\rangle = R_1 |\Phi_p^1\rangle, \quad (6b)$$

where

$$\langle \mathbf{f} | \Phi_p^1 \rangle = \frac{\tilde{q}_{fp}}{2E} \langle \mathbf{f} | \Phi_p^0 \rangle,$$

$$\mathbf{q}_{fp} = \mathbf{f} - \mathbf{p}, \quad \tilde{q} = \alpha \mathbf{q}.$$

We can then obtain for (5a) the expression

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2 \langle \mathbf{k} | \hat{V}_1 | \Phi_p \rangle = 2\pi^2 \{ \langle \mathbf{k} | \hat{V}_1 | \Phi_p^0 \rangle + \langle \mathbf{k} | \hat{V}_1 | \Phi_p^1 \rangle \}. \quad (7)$$

The second term of (7) can be transformed with the aid of (3a) and (6b):

$$\begin{aligned} \langle \mathbf{k} | \hat{V}_1 | \Phi_p \rangle &= \langle \mathbf{k} | \hat{V}_1 R_1 | \Phi_p^1 \rangle = \langle \mathbf{k} | R^\dagger \hat{V}_1 | \Phi_p^1 \rangle \\ &= \langle \Phi_k | \hat{V}_1 | \Phi_p^1 \rangle = \langle \Phi_k^0 | \hat{V}_1 | \Phi_p^1 \rangle + \langle \Phi_k^1 | \hat{V}_1 R_1 | \Phi_p^1 \rangle. \end{aligned} \quad (8)$$

Substituting (8) in (7) we obtain

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2 \{ \langle \mathbf{k} | \hat{V}_1 | \Phi_p^0 \rangle + \langle \Phi_k^0 | \hat{V}_1 | \Phi_p^1 \rangle + \langle \Phi_k^1 | \hat{V}_1 R_1 | \Phi_p^1 \rangle \}. \quad (9)$$

3. CHOICE OF POTENTIALS

As the total electrostatic potential of the nucleus with account of its finite dimensions and screening we can choose a potential in the form

$$V = (V_s - V_C) + V_n, \quad (10)$$

where V_s is the screened potential of the point-like nucleus V_C is the Coulomb potential of the point-like nucleus and V_n is the potential of the nucleus with distributed charge.

Following Moliere,^[9] we use for the first term of (10) in momentum space the expression

$$\langle \mathbf{f} | V_s - V_C | \mathbf{s} \rangle = -\alpha Z \sum_{i=1}^4 a_i / 2\pi^2 (q_{fs}^2 + \lambda_i^2), \quad \lambda_i = \eta b_i, \quad (11)$$

$$a_1 = 0.10, \quad a_2 = 0.55, \quad a_3 = 0.35,$$

$$a_4 = -1, \quad \sum_{i=1}^4 a_i = 0, \quad (11a)$$

$$b_1 = 6.0, \quad b_2 = 1.20, \quad b_3 = 0.30, \quad b_4 = 0,$$

$$\eta = mZ^{1/2}/121, \quad \alpha = e^2 = 1/137. \quad (11b)$$

Using, in analogy with Uberall,^[10] the Laplace transform for the nuclear charge density*

$$\rho(r) = \int_0^\infty \varphi(\lambda) e^{-\lambda r} d\lambda, \quad (12)$$

we can obtain an expression for V_n in the form

$$\begin{aligned} \langle \mathbf{f} | V_n | \mathbf{s} \rangle &= -\frac{\alpha}{2\pi^2} \frac{1}{q^2} \int \rho(r) e^{-i\mathbf{q}r} d^3r \\ &= -\frac{\alpha}{2\pi^2} \frac{1}{q^2} 4\pi \int_0^\infty \frac{\varphi(\lambda)}{q^2 + \lambda^2} d\lambda \\ &= -\frac{\alpha}{2\pi^2} \left\{ \frac{1}{q^2} 4\pi \int_0^\infty \frac{\varphi(\lambda)}{\lambda^2} d\lambda - 4\pi \int_0^\infty \frac{\varphi(\lambda)}{\lambda^2} \frac{1}{q^2 + \lambda^2} d\lambda \right\}, \end{aligned} \quad (13)$$

$\mathbf{q} = \mathbf{f} - \mathbf{s}$.

Recognizing that

$$4\pi \int_0^\infty \frac{\varphi(\lambda)}{\lambda^2} d\lambda = 4\pi \int_0^\infty \rho(r) r^2 dr = Z,$$

substituting (13) and (11) in (10), and choosing for V_1 the Coulomb potential of the nucleus, we obtain

$$\langle \mathbf{f} | V | \mathbf{s} \rangle = \langle \mathbf{f} | V_1 | \mathbf{s} \rangle + \langle \mathbf{f} | V_2 | \mathbf{s} \rangle, \quad (14)$$

$$\langle \mathbf{f} | V_1 | \mathbf{s} \rangle = -V(0), \quad \langle \mathbf{f} | V_2 | \mathbf{s} \rangle = -\gamma_i V(\lambda_i),$$

$$V(\lambda) = \frac{\alpha Z}{2\pi^2} \frac{1}{q_{fs}^2 + \lambda^2}, \quad (14a)$$

$$\gamma_i \equiv \sum_{i=1}^4 a_i - \frac{4\pi}{Z} \int_0^\infty d\lambda_i \frac{\varphi(\lambda_i)}{\lambda_i^2}. \quad (14b)$$

Confining ourselves to a relative accuracy of order $(\alpha Z)^2$, we must include in the amplitude terms up to order $(\alpha Z)^3$ [since the zeroth term, which is the first term in (9), is of order (αZ)]. We can therefore replace R_1 by unity in the third term of (9).

In this part of the paper we confine ourselves to the case when the norm of the operator (14b) satisfies the inequality

$$\alpha Z > \|\gamma\| > (\alpha Z)^2. \quad (15)$$

In the computational accuracy employed it is therefore sufficient to include only the first (linear)

*The screening can, generally speaking, be accounted for in the same manner.

term of (5b). In the second part of the paper we shall consider the case

$$1 > \|\gamma\| > \alpha Z, \quad (15a)$$

when the second term of (5b) must also be included.

Taking (15) into account, we can rewrite the first term of (5b) as

$$f_2(\mathbf{k}, \mathbf{p}) = 2\pi^2 \{ \langle \varphi_{\mathbf{k}}^0 | \hat{V}_2 | \varphi_{\mathbf{p}}^0 \rangle + \langle \varphi_{\mathbf{k}}^0 | \hat{V}_2 | \varphi_{\mathbf{p}}^I \rangle + \langle \varphi_{\mathbf{k}}^I | \hat{V}_2 | \varphi_{\mathbf{p}}^0 \rangle \}. \quad (16)$$

4. CALCULATION OF THE SCATTERING AMPLITUDE

The functions $\varphi_{\mathbf{p}}^0$ and $\varphi_{\mathbf{p}}^I$ in momentum space can be represented in the form^[8]

$$\langle \mathbf{f} | \varphi_{\mathbf{p}}^0 \rangle = -\frac{\partial}{\partial \varepsilon} \Phi(\mathbf{q}, \mathbf{p}, \varepsilon) \Big|_{\varepsilon \rightarrow 0}, \quad \langle \varphi_{\mathbf{p}}^0 | \mathbf{f} \rangle = \langle \mathbf{f} | \varphi_{\mathbf{p}}^0 \rangle, \quad (17a)$$

$$\langle \mathbf{f} | \varphi_{\mathbf{p}}^I \rangle = \frac{\tilde{q}}{2E} \langle \mathbf{f} | \varphi_{\mathbf{p}}^0 \rangle = \alpha Z \frac{1}{-2i\xi} \tilde{\nabla}_p \Phi(\mathbf{q}, \mathbf{p}, 0),$$

$$\langle \varphi_{\mathbf{p}}^I | \mathbf{f} \rangle = -\langle \mathbf{f} | \varphi_{\mathbf{p}}^I \rangle, \quad (17b)$$

$$\Phi(\mathbf{q}, \mathbf{p}, \varepsilon) = \frac{N}{2\pi^2} \frac{1}{2\pi i} \oint_{\xi}^{(0^+, 1^+)} \frac{dx}{x} \left(\frac{-x}{1-x} \right)^{i\xi} \frac{1}{(\mathbf{q} + \mathbf{p}x)^2 - (\rho x + i\varepsilon)^2}; \quad (17c)$$

$$\mathbf{q} = \mathbf{f} - \mathbf{p}, \quad \xi = \alpha ZE / p = \alpha Z / \beta, \quad N = e^{\pi\xi/2} \Gamma(1 - i\xi).$$

The integration contour in (17c) is a closed curve circling about the points 0 and 1 once in counter-clockwise direction. With the aid of (17) we obtain the following expressions for the matrix elements (9) and (16) [see (14)]:

$$2\pi^2 \langle \varphi_{\mathbf{k}}^0 | \hat{V}(\lambda) | \varphi_{\mathbf{p}}^0 \rangle = \alpha Z N^2 \gamma_4 \left(-\frac{\partial}{\partial \varepsilon_2} \right) \left(-\frac{\partial}{\partial \varepsilon_1} \right) H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda) \Big|_{\varepsilon_1, \varepsilon_2 \rightarrow 0}, \quad (18a)$$

$$2\pi^2 \langle \varphi_{\mathbf{k}}^0 | \hat{V}(\lambda) | \varphi_{\mathbf{p}}^I \rangle = (\alpha Z)^2 N^2 \gamma_4 \left(-\frac{\partial}{\partial \varepsilon_2} \right) \left(-\frac{1}{2i\xi_1} \tilde{\nabla}_p \right) H(\mathbf{q}, \mathbf{k}, \mathbf{p}; 0, \varepsilon_2; \lambda) \Big|_{\varepsilon_2 \rightarrow 0}, \quad (18b)$$

$$2\pi^2 \langle \varphi_{\mathbf{k}}^I | \hat{V}(\lambda) | \varphi_{\mathbf{p}}^I \rangle = (\alpha Z)^3 N^2 \gamma_4 \left(-\frac{1}{2i\xi_2} \tilde{\nabla}_k \right) \left(-\frac{1}{2i\xi_1} \tilde{\nabla}_p \right) H(\mathbf{q}, \mathbf{k}, \mathbf{p}; 0, 0; \lambda), \quad (18c)$$

$$H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda) = \frac{1}{(2\pi i)^3} \oint_{x_2}^{(0^+, 1^+)} \frac{dx_2}{x_2} \left(\frac{-x_2}{1-x_2} \right)^{i\xi_2} \oint_{x_1}^{(0^+, 1^+)} \frac{dx_1}{x_1} \left(\frac{-x_1}{1-x_1} \right)^{i\xi_1} \frac{1}{(2\pi^2)^3} \times \int \frac{d^3 f d^3 s}{[(q_{kf} - kx_2)^2 - (kx_2 + i\varepsilon_2)^2] (q_{fs}^2 + \lambda^2) [(q_{sp} + px_1)^2 - (\rho x_1 + i\varepsilon_1)^2]}. \quad (18d)$$

In the appendix we transform the expression for $H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda)$ to the form

$$H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda) = \int_{\varepsilon_2}^{\infty} d\varepsilon_2' \int_{\varepsilon_1}^{\infty} d\varepsilon_1' K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta), \quad (19)$$

$$K(\mathbf{q}, \mathbf{k}, \mathbf{p}; \eta) = \frac{1}{2\pi i} \oint_{\xi}^{(0^+, 1^+)} \frac{dx}{x} \left(\frac{-x}{1-x} \right)^{i\xi_2} \frac{\{a + bx\}^{i\xi_2}}{\{\alpha + \beta x\}} \frac{1}{a + bx}, \quad (19a)$$

where

$$a = q^2 + \eta^2, \quad \alpha = (\mathbf{q} + \mathbf{p})^2 - (p + i\eta)^2,$$

$$b = -2(\mathbf{q}\mathbf{k} + k\mathbf{i}\eta), \quad \beta = -2\{(\mathbf{q} + \mathbf{p})\mathbf{k} + k(p + i\eta)\};$$

$$\eta = \varepsilon_1' + \varepsilon_2' + \lambda. \quad (19b)$$

We note that $\mathbf{q}$ is constant when it comes to differentiation in (18b) and (18c).

The matrix element (18a) enters only in (16), and within the framework of the accuracy employed we should therefore confine ourselves in (18a) and (18b) to terms linear in ξ and in (18c) to the zeroth term. When $\xi_1 = 0$ the value of the integral in (19a) is given by the residue of the last term of the integrand. When $\xi_2 = 0$, the contour of integration can be contracted to zero and the integral becomes equivalent to $\delta(x)$. Taking these circumstances into account, we obtain

$$K(\mathbf{q}, \mathbf{k}, \mathbf{p}; \eta) = \frac{1}{a} \left(1 + 2i\xi \ln \frac{a}{\alpha} \right). \quad (20a)$$

Calculating the gradient with respect to $\mathbf{p}$ under the integral sign in (18b) and then calculating the residues, we obtain

$$-\frac{\tilde{\nabla}_p}{2i\xi_1} K = \frac{1}{\alpha a} \left\{ \tilde{C}_p + i\xi_1 \tilde{C}_p \ln \frac{a}{\alpha} - i\xi_2 \left[\tilde{C}_p \ln(1+B) + \frac{\tilde{C}_p A + \tilde{D}}{A-B} \ln \frac{1+A}{1+B} \right] \right\}, \quad (20b)$$

$$\mathbf{C}_p = \mathbf{q} - i\eta\mathbf{p}/p, \quad \mathbf{D} = \mathbf{k} + k\mathbf{p}/p, \quad A = b/a, \quad B = \beta/\alpha. \quad (20c)$$

Finally, calculating the gradient with respect to $\mathbf{k}$ of (20b), we get

$$\left(-\frac{\tilde{\nabla}_k}{2i\xi_2} \right) \left(-\frac{\tilde{\nabla}_p}{2i\xi_1} \right) K = \frac{1}{2} \frac{1}{\alpha a} \left\{ 2 + \frac{1}{A-B} \ln \frac{1+A}{1+B} \right\} \left(1 - \frac{\tilde{k} \cdot \tilde{p}}{kp} \right). \quad (20d)$$

Taking (20c) and (19b) into account, we substitute (20a), (20b), and (20c) into (18a), (18b), and (18c). Then, recognizing that the first term in (9) is the Schrödinger scattering amplitude in a Coulomb field* divided by $2E$, and taking account of the identities

$$\bar{u}_{\mathbf{k}} \gamma_4 \tilde{k} u_{\mathbf{p}} = \bar{u}_{\mathbf{k}} \gamma_4 \tilde{p} u_{\mathbf{p}} = u_{\mathbf{k}} (\gamma_4 E - m) u_{\mathbf{p}}, \quad (21a)$$

$$\bar{u}_{\mathbf{k}} \gamma_4 \tilde{k} p u_{\mathbf{p}} = \bar{u}_{\mathbf{k}} \gamma_4 (\gamma_4 E - m)^2 u_{\mathbf{p}}, \quad (21b)$$

*We note that the scattering amplitude in a Coulomb field is multiplied by the distorting phase factor $\exp[-i\xi \ln(2pr)]$, but if an infinitesimally small screening is introduced, a factor $\exp[i\xi \ln(2p/\lambda)]$ appears in front of the entire wave function (λ is the screening parameter).^[8] These two factors cancel each other.

we obtain the following expressions for the amplitudes (9) and (10)

$$f_1(\mathbf{k}, \mathbf{p}) = \gamma_4 \frac{\alpha Z}{q^2} \left\{ \exp(i\xi 2 \ln \varepsilon) + \alpha Z M \frac{E - \gamma_4 m}{p} K_1(0) \right. \\ \left. + (\alpha Z)^2 \frac{M}{p^2} [2E(E - \gamma_4 m) K_2(0) + m(\gamma_4 E - m) K_3(0)] \right\} e^{i\varphi}, \quad (22)$$

$$f_2(\mathbf{k}, \mathbf{p}) = \gamma_4 \frac{\alpha Z}{q^2} \gamma_i M \left\{ \frac{\varepsilon^2}{\varepsilon^2 + \mu_i^2} \right. \\ \left. + 2\alpha Z \left[i \frac{E}{p} \frac{\varepsilon^2}{\varepsilon^2 + \mu_i^2} \ln \frac{\varepsilon^2 + \mu_i^2}{-i\mu_i(1 + i\mu_i)} \right. \right. \\ \left. \left. + \frac{E - \gamma_4 m}{p} K_1(\mu_i) \right] \right\} e^{i\varphi}, \quad (23)$$

where

$$\varepsilon = \frac{q}{2p} = \sin \frac{\vartheta}{2}, \quad \mu_i = \frac{\lambda_i}{2p},$$

$$M = -N^2 e^{-i\varphi} = 2\pi\xi / (1 - e^{-2\pi\xi}),$$

$$e^{i\varphi} = -\frac{\Gamma(1 - i\xi)}{\Gamma(1 + i\xi)},$$

and γ_i is given by (14b). Furthermore,

$$K_1(\mu) = \varepsilon^2 \int_{\mu}^{\infty} \frac{d\eta}{(1 + i\eta)(\varepsilon^2 + \eta^2)} \\ = \frac{i\varepsilon}{2} \left(\frac{1}{1 - \varepsilon} \ln \frac{\varepsilon + i\mu}{1 + i\mu} - \frac{1}{1 + \varepsilon} \ln \frac{-\varepsilon + i\mu}{1 + i\mu} \right), \quad (24)$$

$$K_2(\mu) = \varepsilon^2 \int_{\mu}^{\infty} \frac{id\mu}{(1 + i\eta)(\varepsilon^2 + \eta^2)} \ln \frac{\varepsilon^2 + \eta^2}{-i\eta(1 + i\eta)} \\ = \frac{\varepsilon^2}{1 - \varepsilon^2} \left\{ L_2\left(\frac{1 + \varepsilon}{1 + i\mu}\right) + L_2\left(\frac{1 - \varepsilon}{1 + i\mu}\right) - L_2\left(\frac{1}{1 + i\mu}\right) \right\} \\ + \frac{1}{2\varepsilon} \left\{ \frac{1}{1 - \varepsilon} L_2\left(\frac{\varepsilon}{\varepsilon + i\mu}\right) - \frac{1}{1 + \varepsilon} L_2\left(\frac{\varepsilon}{\varepsilon - i\mu}\right) \right. \\ \left. + \frac{1}{1 - \varepsilon} L_2\left(\frac{\varepsilon - 1}{\varepsilon + i\mu}\right) - \frac{1}{1 + \varepsilon} L_2\left(\frac{1 + \varepsilon}{\varepsilon - i\mu}\right) - \frac{1}{1 - \varepsilon} L_2\left(\frac{2\varepsilon}{\varepsilon + i\mu}\right) \right. \\ \left. + \frac{1}{1 + \varepsilon} L_2\left(\frac{2\varepsilon}{\varepsilon - i\mu}\right) \right\}, \quad (25)$$

$$K_3(\mu) = -\int_{\mu}^{\infty} \eta d\eta \left(\frac{2\varepsilon^2}{i\eta(1 + i\eta)(\varepsilon^2 + \eta^2)} + \frac{1}{(1 + i\eta)^2} \ln \frac{\varepsilon^2 + \eta^2}{\eta^2} \right) \\ = 2L_2\left(\frac{1}{1 + i\mu}\right) - L_2\left(\frac{1 + \varepsilon}{1 + i\mu}\right) - L_2\left(\frac{1 - \varepsilon}{1 + i\mu}\right). \quad (26)^*$$

The integrals (24) and (26) are calculated by making the substitution

$$i\eta \rightarrow t, \quad \ln \frac{t + b}{t + a} = \int_a^b \frac{dx}{x + t},$$

where the integration along the imaginary axis in the complex t plane is replaced by integration along the real axis with all the singularities circumscribed from above. The function $L_2(z)$ contained in (25) and (26) is the logarithmic Euler function^[11] (see also^[5]):

*The single integral (26) is obtained from the double integral (18c) by integration by parts.

$$L_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt = \sum_{n=0}^{\infty} \frac{z^n}{n^2}. \quad (27)$$

All the multiple-valued functions in (24) - (26) are specified in terms of their principal values on a plane with a cut from $z = 1$ to $z = +\infty$ along the positive axis, where $2\pi > \arg(z - 1) > 0$.

The function $L_2(z)$ has the following properties, which make it readily possible to calculate its imaginary and real parts

$$L_2(z) + L_2\left(\frac{1}{z}\right) = \pi^2/3 - \frac{1}{2} \ln^2 z + \pi i \ln z, \quad (28a)$$

$$L_2(1 - z) + L_2(z) = \pi^2/6 - \ln z \ln(1 - z), \quad (28b)$$

$$\frac{1}{2} L_2(z^2) = L_2(z) + L_2(-z), \quad (28c)$$

$$\operatorname{Im} L_2(z) = 0 \quad \text{for} \quad -1 \leq z \leq 1. \quad (28d)$$

5. SCATTERING CROSS SECTION IN THE PRESENCE OF INITIAL POLARIZATION

The cross section in the presence in the initial polarization ξ of the incoming particle is obtained from (1) with the aid of the formula^[7]

$$\sigma(\vartheta) = \frac{1}{2} \operatorname{Sp} f(\mathbf{k}, \mathbf{p}) (1 - \hat{a} \gamma_5) (m - \hat{p}) \bar{f}(\mathbf{k}, \mathbf{p}) (m - \hat{i} \hat{k}), \\ \bar{f} = \gamma_4 f^\dagger \gamma_4, \quad \mathbf{a} = \boldsymbol{\zeta} + \frac{p \boldsymbol{\zeta}}{m(m + E)} \mathbf{p}, \quad a_0 = \frac{p \zeta}{m}; \quad (29)$$

The expression for $f(\mathbf{k}, \mathbf{p})$ is given by (5), (22), and (23).

After calculating the trace in (29) and transform (25) and (26) with the aid of (28a) and (28b), we obtain the following expression for the cross section:

$$\sigma = \sigma_R \{Q_1 + Q_2 + (R_1 + R_2) n \boldsymbol{\zeta}\}, \\ \sigma_R = \left(\frac{\xi}{2p} \right)^2 \frac{1}{\varepsilon^4}, \quad \mathbf{n} = \frac{[\mathbf{K} \mathbf{p}]}{\sin \vartheta}, \quad (30)^*$$

$$Q_1 = 1 - \beta^2 \varepsilon^2 + \alpha Z \pi M \beta \varepsilon (1 - \varepsilon) + 2(\alpha Z)^2 M \varepsilon \{L_2(\varepsilon) \\ - L_2(-\varepsilon) + \ln \varepsilon \left(\ln \frac{1 - \varepsilon}{1 + \varepsilon} + \varepsilon \ln \varepsilon \right) + \frac{\pi^2}{6} (2\varepsilon - 3) \\ - \beta^2 \varepsilon [L_2(\varepsilon) + L_2(-\varepsilon) \\ + \ln \varepsilon \left(\ln(1 - \varepsilon^2) - \frac{M}{2} \frac{\varepsilon^2}{1 - \varepsilon^2} \ln \varepsilon \right) - \frac{M}{2} \frac{\pi^2}{4} \frac{1 - \varepsilon}{1 + \varepsilon} \} \}, \quad (31)$$

$$R_1 = -2\alpha Z M \varepsilon^3 \left(\frac{1 - \beta^2}{1 - \varepsilon^2} \right)^{1/2} \left\{ \beta \ln \varepsilon + \pi \alpha Z \left[\frac{2 \ln 2}{\varepsilon} \right. \right. \\ \left. \left. - \frac{1 + \varepsilon}{\varepsilon^3} \ln(1 + \varepsilon) \right] \right\}, \quad (32)$$

$$Q_2 = 2M \gamma_i \left\{ \frac{\varepsilon^2}{\varepsilon^2 + \mu_i^2} (1 - \beta^2 \varepsilon^2) + \alpha Z \left[\frac{\varepsilon^2}{\varepsilon^2 + \mu_i^2} (1 - \beta^2 \varepsilon^2) \frac{1}{\beta} \right. \right. \\ \left. \left. \times \left(\pi \frac{M}{2} \frac{\varepsilon}{1 + \varepsilon} - 2 \operatorname{arctg} \frac{1}{\mu_i} \right) + \beta \varepsilon \left((1 - \varepsilon) \operatorname{arctg} \frac{\mu_i(1 + \varepsilon)}{-\varepsilon + \mu_i^2} \right. \right. \right. \\ \left. \left. \left. - (1 + \varepsilon) \operatorname{arctg} \frac{\mu_i(1 - \varepsilon)}{\varepsilon + \mu_i^2} \right) \right] \right\}. \quad (33)^\dagger$$

* $[\mathbf{k} \mathbf{p}] = \mathbf{k} \times \mathbf{p}$.

$\dagger \operatorname{arctg} = \tan^{-1}$.

$$R_2 = -2\alpha Z M \gamma_i \beta (1 - \beta^2)^{1/2} \frac{\varepsilon^3}{(1 - \varepsilon^2)^{1/2}} \ln \frac{\varepsilon^2 + \mu_i^2}{1 + \mu_i^2}; \quad (34)$$

$$\gamma_i \equiv \sum_{i=1}^4 a_i - \frac{4\pi}{Z} \int_0^\infty d\mu_i \frac{\varphi(2p\mu_i)}{2p\mu_i^2} \quad (\text{cf. (14) and (11)}),$$

$$\varepsilon = \frac{q}{2p} = \sin \frac{\theta}{2}, \quad \beta = \frac{p}{E}, \quad M = \frac{2\pi\xi}{1 - e^{-2\pi\xi}},$$

$$\xi = \frac{\alpha Z E}{p} = \frac{\alpha Z}{\beta}, \quad \mu_i = \frac{\lambda_i}{2p}.$$

In the extreme relativistic case ($\beta \rightarrow 1$, $p \rightarrow \infty$), the screening potential drops out as a result of (11a), and (31) – (34) assume the form

$$Q_1^{\text{cr}} = 1 - \varepsilon^2 + \alpha Z \pi \varepsilon (1 - \varepsilon) + 2(\alpha Z)^2 \varepsilon \left\{ (1 - \varepsilon) L_2(\varepsilon) - (1 + \varepsilon) L_2(-\varepsilon) + \ln \varepsilon \left[(1 - \varepsilon) \ln(1 - \varepsilon) - (1 + \varepsilon) \ln(1 + \varepsilon) + \frac{\varepsilon(2 - \varepsilon^2)}{2(1 - \varepsilon^2)} \ln \varepsilon \right] - \frac{\pi^2}{24} \frac{\varepsilon(1 + 7\varepsilon)}{1 + \varepsilon} \right\}, \quad (35)$$

$$Q_2^{\text{cr}} = 2\gamma \left\{ \frac{\varepsilon^2}{\varepsilon^2 + \mu^2} + \alpha Z \left[\frac{\varepsilon^2}{\varepsilon^2 + \mu^2} \left(\pi \frac{3\varepsilon + 2}{2(1 + \varepsilon)} - 2 \operatorname{arctg} \frac{1}{\mu} \right) + \varepsilon \left((1 - \varepsilon) \operatorname{arctg} \frac{\mu(1 + \varepsilon)}{-\varepsilon + \mu^2} - (1 + \varepsilon) \operatorname{arctg} \frac{\mu(1 - \varepsilon)}{\varepsilon + \mu^2} \right) \right] \right\}; \quad (36)$$

$$\gamma = -\frac{4\pi}{Z} \int_0^\infty d\mu \frac{\varphi(2p\mu)}{2p\mu^2}, \quad R_1^{\text{cr}} = R_2^{\text{cr}} = 0. \quad (37)$$

In conclusion, the author thanks Prof. L. A. Sliv for continuous support during the work.

APPENDIX

Using the formula (see [3] and [8])

$$\begin{aligned} & \frac{1}{2\pi^2} \int \frac{d^3s}{(q_{ns}^2 - \Lambda^2) [(q_{sp} + r)^2 - (r + i\eta)^2]} \\ &= \frac{i}{2|q_{np} + r|} \ln \frac{|q_{np} + r| + r + \Lambda + i\eta}{|q_{np} + r| + r + \Lambda + i\eta} \\ &= \int_{\eta}^{\infty} \frac{d\lambda}{(q_{np} + r)^2 - (r + \Lambda + i\lambda)^2}, \end{aligned}$$

We transform the three-dimensional integrals in (18d), after which we get

$$\begin{aligned} H(q, k, p; \varepsilon_1, \varepsilon_2; \lambda) &= \frac{1}{(2\pi i)^3} \oint_{\varepsilon_2}^{(0+, 1+)} \frac{dx_2}{x_2} \left(\frac{-x_2}{1 - x_2} \right)^{i\varepsilon_2} \oint_{\varepsilon_1}^{(0+, 1+)} \frac{dx_1}{x_1} \left(\frac{-x_1}{1 - x_1} \right)^{i\varepsilon_1} \\ &\times \int_{\varepsilon_2}^{\infty} d\varepsilon_2' \int_{\varepsilon_1}^{\infty} d\varepsilon_1' \frac{1}{(q + px_1 - kx_2)^2 - (kx_2 + px_1 + i\eta)^2}; \end{aligned} \quad (A.2)$$

$$q = k - p, \quad \eta = \varepsilon_1 + \varepsilon_2' + \lambda.$$

By evaluating the integral with respect to x_1 by residues and putting $x = x_2$, we obtain (19).

¹ N. F. Mott, Proc. Roy. Soc. **A124**, 425 (1929); **A135**, 429 (1932).

² W. A. McKinlay and H. Feshbach, Phys. Rev. **74**, 1759 (1948).

³ R. H. Dalitz, Proc. Roy. Soc. **A206**, 509 (1951).

⁴ H. Mitter and P. Urban, Acta Phys. Austriaca, **7**, 311 (1953).

⁵ C. Kacser, Nuovo cimento **13**, 303 (1959).

⁶ Johnson, Weber, and Mullin, Phys. Rev. **121**, 933 (1961).

⁷ A. I. Akhiezer and V. B. Berestetskii, Kvantovaya élektrodinamika (Quantum Electrodynamics), 2d Ed. Fizmatgiz, 1959.

⁸ V. G. Gorshkov, JETP **40**, 1481 (1961), Soviet Phys. JETP **13**, 1037 (1961).

⁹ G. Moliere, Z. Naturforsch. **2a**, 133 (1947).

¹⁰ H. Uberall, Phys. Rev. **119**, 365 (1960).

¹¹ H. Bateman, Higher Transcendental Functions, **1**, McGraw-Hill, (1953) p. 31.

(A.1)

Translated by J. G. Adashko
168

Letters to the Editor

COSMIC-RAY EQUATOR ACCORDING TO THE DATA FROM THE SECOND SOVIET SPACESHIP

I. A. SAVENKO, P. I. SHAVRIN, V. E. NESTEROV,
and N. F. PISARENKO

Submitted to JETP editor June 7, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 985-986
(September, 1961)

THE knowledge of the geographical position of the line of minimum primary cosmic-ray intensity (cosmic-ray equator) permits a study of the character and structure of the geomagnetic field and a check of the theoretical and empirical approximations of this field.

The use of satellites for the determination of the cosmic-ray equator has a number of advantages over earth-based investigations: 1) a large number of crossings of the equator at various points, 2) the practically simultaneous crossing of the equator over the whole globe, 3) a direct detection of the primary component of cosmic radiation. These advantages provide the possibility of studying in detail the cosmic-ray equator at various periods of time, and in particular of studying the influence of various geophysical effects on its position. It becomes unnecessary to introduce barometric and temperature corrections, or corrections for time variations.

The ionizing-radiation detectors on the second spaceship included a gas-discharge counter. The pulses from the counter were fed to a scaler which was interrogated by an independent daily memory

device once every three minutes. This information was recalled and transmitted to the earth stations by means of telemetry controlled from the earth. The daily memory made it possible to measure the latitude dependence of the primary cosmic radiation for each crossing of the equator. Since, at high latitudes, the spaceship often passed through radiation belts, we have used only the experimental points for latitudes lower than 40° in constructing the empirical formula for the latitude dependence by the least-squares method. A quadratic parabola was used as the approximating function.

From 22 latitude curves obtained at different crossings of the geographic equator region, we have determined the position of the minima of cosmic-ray intensity (see figure).

The cosmic-ray equator obtained is not compatible with the idea of a dipole geomagnetic field.^[1-6] The comparison of the cosmic-ray equator with the equator calculated by Quenby and Webber taking into account both the dipole and nondipole components of the geomagnetic field,^[7] and also with the equator calculated by Kellogg and Schwartz using the octupole approximation,^[8] gives a sufficiently good agreement within the limits of experimental accuracy. A more detailed comparison would be possible if the experimental errors could be decreased.

The authors would like to thank S. N. Vernov, N. L. Grigorov, and I. P. Ivanenko for discussing the results.

¹ Simpson, Fenton, and Rose, *Phys. Rev.* **102**, 1648 (1956).

² Katz, Meyer, and Simpson, *Nuovo cimento* **10**, Suppl. 8, 277 (1958).

³ P. Rothwell and J. Quenby, *Nuovo cimento* **10**, Suppl. 249 (1958).



Cosmic-ray equator. Dash-dot line — geomagnetic equator of the dipole field, dotted line — equator calculated taking into account the dipole and nondipole components of the internal geomagnetic field,^[7] solid-broken line — equator calculated using the octupole approximation,^[8] ● — data obtained by the second spaceship.

⁴J. R. Storey, Phys. Rev. **113**, 297 (1959).

⁵Pomerantz, Sandström, Potnis, and Rose, Proc. Cosmic Ray Conf. IUPAP, Moscow, 1959, vol. IV.

⁶Pomerantz, Potnis, and Sandström, J. Geophys. Research **65**, 3539 (1960).

⁷J. J. Quenby and W. R. Webber, Phil. Mag. **4**, 90 (1959).

⁸P. J. Kellog and M. Schwartz, Nuovo cimento **13**, 761 (1959).

Translated by H. Kasha

169

NONLINEAR PROPERTIES OF THREE-LEVEL SYSTEMS

V. M. FAÏN, Ya. I. KHANIN, and É. G. YASHCHIN

Radiophysics Institute, Gorkii State University

Submitted to JETP editor June 26, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 986-988 (September, 1961)

AS is well known, the three-level system* is the basic element of quantum amplifiers and oscillators-masers. It is of interest to consider other possible physical applications of three-level systems, in particular, applications that derive from their nonlinear properties.

A manifestation of the nonlinear properties of a three-level system would be the response of the system (for example, the polarization P) to two monochromatic signals. Let E_1 , E_2 and E_3 be the three levels of the quantum system and suppose that an external field (electric or magnetic) acts on the system

$$F = E_{13} \cos \Omega_{31}t + E_{23} \cos \Omega_{32}t, \quad (1)$$

where $\Omega_{31} \approx (E_3 - E_1)/\hbar$ and $\Omega_{32} \approx (E_3 - E_2)/\hbar$.

To find the system polarization produced by the field (1) we use the equation for the density matrix ρ_{mn} :^[1,2]

$$\begin{aligned} \frac{\partial \rho_{mn}}{\partial t} + i\omega_{mn}\rho_{mn} &= -\frac{i}{\hbar} F \sum_{l=1}^3 (\mu_{ml}\rho_{ln} - \rho_{ml}\mu_{ln}) - [\tau^{-1}(\rho - \rho_0)]_{mn}; \\ [\tau^{-1}(\rho - \rho_0)]_{mn} &= \begin{cases} \tau_1^{-1}(\rho - \rho_0)_{mn} & \text{for } m=n \\ \tau_2^{-1}\rho_{mn} & \text{for } m \neq n \end{cases}, \end{aligned} \quad (2)$$

where the μ_{ml} are the dipole moment matrix elements, τ_1 and τ_2 are the longitudinal and trans-

verse relaxation times, and ρ_{0mn} is the density matrix corresponding to instantaneous equilibrium at time t , when the field is given by $F(t)$. The polarization of the system is

$$P = \text{Sp}(\hat{\rho}\hat{\mu}). \quad (3)$$

In solving Eq. (2) we keep only the resonance terms^[2] at frequencies Ω_{32} , Ω_{31} , and $\Omega_{31} - \Omega_{32}$, thereby obtaining the corresponding system of algebraic equations^[2] that yields the following expression:

$$P = \rho_{31}\mu_{13}e^{-i\Omega_{31}t} + \rho_{32}\mu_{23}e^{-i\Omega_{32}t} + \rho_{21}\mu_{12}e^{-i(\Omega_{31}-\Omega_{32})t} + \text{c.c.}, \quad (4)$$

if $\Omega_{31} = (E_3 - E_1)/\hbar$ and $\Omega_{32} = (E_3 - E_2)/\hbar$,

$$\rho_{31} = 2i\gamma_{31}\Delta^{-1}\{D_{13}^{(0)}[4(\tau_2^{-1} + \gamma_{23}^2\tau_1) + \tau_2\gamma_{13}^2]$$

$$- D_{23}^{(0)}(2\tau_1 + \tau_2)\gamma_{13}^2\},$$

$$\rho_{32} = 2i\gamma_{32}\Delta^{-1}\{D_{23}^{(0)}[4(\tau_2^{-1} + \gamma_{13}^2\tau_1) + \tau_2\gamma_{23}^2]$$

$$- D_{13}^{(0)}(2\tau_1 + \tau_2)\gamma_{13}^2\},$$

$$\rho_{21} = \frac{1}{2}i\tau_2\gamma_{13}\gamma_{23}(\rho_{32}/\gamma_{32} + \rho_{31}/\gamma_{13})$$

$$= -2\gamma_{13}\gamma_{23}\tau_2\Delta^{-1}\{D_{13}^{(0)}[2(\tau_2^{-1} + \tau_1\gamma_{23}^2) - \tau_1\gamma_{13}^2]$$

$$+ D_{23}^{(0)}[2(\tau_2^{-1} + \gamma_{13}^2\tau_1) - \tau_1\gamma_{23}^2]\};$$

$$\Delta = [4(\tau_2^{-1} + \gamma_{23}^2\tau_1) + \tau_2\gamma_{13}^2][4(\tau_2^{-1} + \gamma_{13}^2\tau_1)$$

$$+ \tau_2\gamma_{23}^2] - (2\tau_1 + \tau_2)^2\gamma_{13}^2\gamma_{23}^2;$$

$$\gamma_{13} = \mu_{13}E_{13}/\hbar = \gamma_{31}, \quad \gamma_{23} = \mu_{23}E_{23}/\hbar = \gamma_{32};$$

where $D_{13}^{(0)}$ and $D_{23}^{(0)}$ are the corresponding equilibrium population differences in the levels.

It is evident from Eq. (4) that the response of the system to two monochromatic signals contains a term at the combination frequency $\Omega_{12} = \Omega_{13} - \Omega_{23}$; this term results from the nonlinearity of the system.[†] In particular, in the case of practical interest, where one of the applied signals is large (for example, $\gamma_{13}\tau_2 > 1$), it turns out that $|\rho_{12}|/|\rho_{23}| > 1$. This result indicates the possibility of building a quantum frequency converter with appreciable conversion gain. From the point of view of noise characteristics it would appear that the quantum converter can compete with quantum amplifiers. Special interest attaches to the case in which Ω_{12} is a very low frequency and can be used directly as the intermediate frequency.

We note that in principle it is possible for oscillations to occur at the frequencies Ω_{13} and Ω_{23} when a three-level maser is used as an oscillator or as an amplifier. It then follows that a maser can produce a signal at frequency Ω_{12} even if the populations in levels E_1 and E_2 are not inverted.

The principle of quantum frequency conversion can also be used in the optical region. It would appear that the monochromatic light signals re-

quired for this purpose could be generated by lasers. In this connection we note that lasers often generate several frequencies rather than a single frequency.^[7,8] One expects that combination frequencies would be generated under these conditions.

*The idealization in which we consider only three levels of the entire system of levels is completely justified since the other levels can be neglected in most cases.

†The Bloembergen equations,^[3] in which the off-diagonal elements of the density matrix are not considered, are frequently used in maser calculations. It will be evident that these equations can not give the effect considered here nor a number of other features of maser operation. The reference made by Bloembergen^[4] to the calculation by Clogston,^[5] in which off-diagonal terms in the density matrix are included, is not well taken because the latter work^[5] contains an error in that the author has not taken account of all the resonance terms in the density matrix. A similar error appears in a paper by one of the present authors.^[6]

¹R. Karplus and J. Schwinger, *Phys. Rev.* **73**, 1020 (1948).

²A. N. Malakhov and V. M. Faïn, *Izv. Vuzov, Radiofizika* (News of the Colleges, Radiophysics) **1**, 66 (1958).

³N. Bloembergen, *Phys. Rev.* **104**, 324 (1956).

⁴N. Bloembergen and S. Shapiro, *Phys. Rev.* **116**, 1453 (1959).

⁵A. Clogston, *J. Phys. Chem. Solids* **4**, 271 (1958).

⁶V. M. Faïn, *JETP* **33**, 1290 (1957), *Soviet Phys. JETP* **6**, 991 (1958).

⁷P. P. Sorokin and M. J. Stevenson, *Phys. Rev. Letters* **5**, 557 (1960).

⁸Javan, Bennett, and Herriott, *Phys. Rev. Letters* **6**, 106 (1961).

Translated by H. Lashinsky
170

NEGATIVE CONDUCTIVITY IN INDUCED TRANSITIONS

N. G. BASOV, O. N. KROKHIN, L. M. LISITSYN,
E. P. MARKIN, and B. D. OSIPOV

P. N. Lebedev Physics Institute, Academy of
Sciences, U.S.S.R.

Submitted to JETP editor July 13, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 988-989
(September, 1961)

A number of methods for producing negative temperatures in semiconductors^[1-3] have been proposed in recent years.

It has been noted^[3] that in indirect transitions the concentration of carriers for which a negative temperature is produced with respect to interband transitions is relatively small, being several orders of magnitude smaller than the concentration at which a negative absorption coefficient is obtained for photons with energies comparable to the width of the forbidden band. In order to obtain a negative absorption coefficient the probability for induced emission of photons in an interband transition must be appreciably greater than the probability of photon absorption in the inverse process, so as to compensate for absorption in internal transitions. However, internal absorption processes have essentially no effect on conductivity since they cause no change in the total number of free carriers; on the other hand, in-

duced interband transitions due to photons incident on a semiconductor in a negative temperature state reduce the number of free current carriers and cause a reduction in conductivity.

Thus, a semiconductor in a negative temperature state with respect to an interband transition should exhibit a negative photoconductivity when irradiated by photons with energies approximately equal to the width of the forbidden band. Measurement of the spectral dependence of the photoconductivity of a semiconductor should allow us to find negative temperature states even if a negative absorption coefficient is not produced.

We have carried out experiments to produce and observe negative temperatures in silicon. Samples maintained at a temperature of 4° K were irradiated by light at wavelengths smaller than 0.7 μ and exhibited an appreciable increase in conductivity. When irradiated by weak monochromatic light in a narrow wavelength band close to 1.1 μ , however, a number of samples exhibited reduced conductivity (negative photoconductivity).

It can be assumed that this reduction in conductivity is due to a negative temperature state, although we have not completely ruled out other explanations for the observed effect, for example, impurity photoconductivity.

¹Basov, Vul, and Popov, *JETP* **37**, 587 (1959), *Soviet Phys. JETP* **10**, 416 (1960).

²Basov, Krokhin, and Popov, *JETP* **38**, 1001 (1960), *Soviet Phys. JETP* **11**, 720 (1960).

³Basov, Krokhin, and Popov, JETP 40, 1203 (1961), Soviet Phys. JETP 13, 845 (1961).

Translated by H. Lashinsky

171

WHAT IS HEAVIER, "MUONIUM ONE" OR "MUONIUM TWO"?

L. B. OKUN' and B. PONTECORVO

Joint Institute for Nuclear Research

Submitted to JETP editor August 10, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 989-991 (September, 1961)

SOME years ago it was noted^[1] that muonium [the atomic system $M \equiv (\mu^+ e^-)$] can spontaneously transmute in vacuum into antimuonium $\bar{M} \equiv (\mu^- e^+)$. The oscillations $M \rightleftharpoons \bar{M}$ would be analogous to the transmutation $K^0 \rightleftharpoons \bar{K}^0$.^[2]

Recently several papers devoted to this problem have appeared in the literature.^[3-6] The aim of the present paper is to emphasize that the analogy between the oscillations $M \rightleftharpoons \bar{M}$ and $K^0 \rightleftharpoons \bar{K}^0$ is even more complete than noted earlier in that the decay of the states which are even and odd in combined parity (i.e., under PC), viz., $M_1 = (M + \bar{M})/\sqrt{2}$ and $M_2 = (M - \bar{M})/\sqrt{2}$, proceeds via different channels, as in the case of K_1^0 and K_2^0 . Here M_1 and M_2 are the muonium states which are stationary in vacuo.

Let us investigate the case where only one kind of neutrino exists and there is no direct $(\mu e)(\mu e)$ interaction.^[1] One would expect this case to correspond to reality if there would hold in nature the so called "Kiev symmetry," i.e., invariance of all weak interaction processes under the substitution $\mu \rightarrow \Lambda$, $e \rightarrow n$, $\nu \rightarrow p$. If $K^0 \rightleftharpoons \bar{K}^0$ oscillations exist this symmetry points to a possibility of $M \rightleftharpoons \bar{M}$ oscillations. In any case the transmutation would in this case be due to the same interaction which is responsible to the decay of the free muon $\mu^+ \rightarrow e^+ + \nu + \bar{\nu}$. Naturally the question arises: in what respect do the even and odd muonium states differ? The decay channels of the odd state M_2 will be

$$e_{fast}^+ + \nu + \bar{\nu} + e_{slow}^- \quad (1)$$

$$e_{fast}^- + \nu + \bar{\nu} + e_{slow}^+ \quad (2)$$

$$\nu + \bar{\nu} \quad (3)$$

We are considering here muonium with spin 1, since a system with spin 0 can not decay into a neutrino pair in view of the neutrino helicity. The even system M_1 , also with spin 1, can decay via channels (1) and (2) but the decay via channel (3) is forbidden. The circumstance that the spin-1 even system can not decay into the pair $\nu + \bar{\nu}$ is similar to the case of spin-0 odd K_2^0 meson which can not decay into two π mesons. By means of Lehman's theorem^[8] one can show that the mass of M_2 is greater than that of M_1 . As is well known, one cannot decide the question of which is heavier, the K_1^0 or the K_2^0 meson, on theoretical grounds,^[3] because of the difficulties associated with the strong interactions.

In contrast to the $K_1^0 - K_2^0$ case, the difference in the decay characteristics of the systems M_1 and M_2 is rather small. Physically this is associated with the large size of the atomic system: even though strictly speaking the decaying objects are M_1 and M_2 , in the overwhelming number of cases it is the "free" muon which decays in the atomic system. In principle, however a difference still exists between the decay channels of M_1 and M_2 . It was thought worthwhile to point this out even if only for pedagogic reasons.

The above arguments about the difference in the decay channels of M_1 and M_2 remain true also if there exists a direct $(\mu e)(\mu e)$ interaction.^[1] However, then the mass difference obviously will be determined by the $(\mu e)(\mu e)$ interaction^[10] and we then will not be able to say anything about the sign of the mass difference of M_1 and M_2 .

We now assume that there exist in nature two neutrino types: ν_e and ν_μ .^[11] If e and ν_e on the one hand and μ and ν_μ on the other hand have different additive quantum numbers (charges) then the transition $M \rightleftharpoons \bar{M}$ is strictly forbidden and it makes no physical sense to talk about M_1 and M_2 .

We now shall discuss the recently proposed^[12,13] possibility that there might exist multiplicative quantum numbers. In that case the decay of the free meson is given by

$$\mu^+ \rightarrow \begin{cases} e^+ + \nu_e + \bar{\nu}_\mu \\ e^+ + \bar{\nu}_e + \nu_\mu \end{cases}$$

and the transmutation $M \rightleftharpoons \bar{M}$ is due to the direct $(\mu e)(\mu e)$ interaction. Then there does not exist a difference in the decay channels of the even and odd muonium states. Both M_1 and M_2 can decay through the channels

$$\begin{aligned} e_{fast}^+ + \nu_e + \bar{\nu}_\mu + e_{slow}^-, & e_{fast}^- + \nu_e + \bar{\nu}_\mu + e_{slow}^+ \\ e_{fast}^+ + \bar{\nu}_e + \nu_\mu + e_{slow}^-, & e_{fast}^- + \bar{\nu}_e + \nu_\mu + e_{slow}^+ \\ \nu_e + \bar{\nu}_\mu, & \nu_e + \nu_\mu. \end{aligned}$$

The authors are grateful to S. S. Gershtein for interesting discussions.

¹B. Pontecorvo, JETP **33**, 549 (1957), Soviet Phys. JETP **6**, 431 (1958).

²M. Gell-Mann and A. Pais, Phys. Rev. **97**, 1387 (1955).

³S. Weinberg and G. Feinberg, Phys. Rev. Lett., in press.

⁴N. Gabbibo and R. Gatto, Nuovo cimento **19**, 612 (1961).

⁵S. L. Glashow, Phys. Rev. Lett. **6**, 196 (1961).

⁶S. L. Glashow, Nuovo cimento **20**, 591 (1961).

⁷Gamba, Marshak, and Okubo, Proc. Nat. Acad. Sci. **45**, 881 (1959).

⁸H. Lehmann, Nuovo cimento **11**, 342 (1954).

⁹I. Yu. Kobzarev and L. B. Okun', JETP **39**, 605 (1960), Soviet Phys. JETP **12**, 426 (1961).

¹⁰L. B. Okun' and B. Pontecorvo, JETP **32**, 1587 (1957), Soviet Phys. JETP **5**, 1297 (1957).

¹¹B. Pontecorvo, JETP **37**, 1751 (1959), Soviet Phys. JETP **10**, 1236 (1960).

¹²R. Gatto, Proc. 1960 Ann. Int. Conf. High Energy Phys., Rochester, Un. of Rochester (1960), p. 609.

¹³G. Feinberg and S. Weinberg, Phys. Rev. Lett. **6**, 381 (1961).

Translated by M. Danos

172

SOVIET PHYSICS JOURNALS

Translations of the 1961 Originals—Published in English by the American Institute of Physics with the cooperation of the National Science Foundation.

Publication year July, 1961—June, 1962.

Soviet Physics—JETP

A translation, beginning with 1955 issues of *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* of the USSR Academy of Sciences. Leading physics journal of Soviet Union. Similar to "The Physical Review" in quality and range of topics. Outstanding new work is most likely to appear in this journal.

Vols. 13 and 14 comprising twelve issues, approx. 4000 pp.
\$75 domestic, \$79 foreign
Libraries* \$50 domestic, \$54 foreign. Single copies, \$8

Soviet Physics—SOLID STATE

A translation, beginning with 1959 issues of *Fizika Tverdogo Tela* of the USSR Academy of Sciences. Offers results of theoretical and experimental investigations in the physics of semiconductors, dielectrics, and on applied physics associated with these problems. Also publishes papers on electronic processes taking place in the interior and on the surface of solids.

Vol. 3 comprising twelve issues, approx. 3800 pp.
\$70 domestic, \$74 foreign
Libraries* \$45 domestic, \$49 foreign. Single copies, \$8

Soviet Physics—TECHNICAL PHYSICS

A translation, beginning with 1956 issues of *Zhurnal Tekhnicheskoi Fiziki* of the USSR Academy of Sciences. Contains work on plasma physics and magnetohydrodynamics, aerodynamics, ion and electron optics, and radio physics. Also publishes articles in mathematical physics, the physics of accelerators, and molecular physics.

Vol. 6 comprising twelve issues, approx. 2000 pp.
\$55 domestic, \$59 foreign
Libraries* \$35 domestic, \$39 foreign. Single copies, \$5

Soviet Physics—ACOUSTICS

A translation, beginning with 1955 issues of *Akusticheskii Zhurnal* of the USSR Academy of Sciences. Devoted principally to physical acoustics but includes electro-, bio-, and psychoacoustics. Mathematical and experimental work with emphasis on pure research.

Vol. 7 comprising four issues, approx. 500 pp.
\$12 domestic, \$14 foreign
(No library discounts.) Single copies, \$4

Soviet Physics—DOKLADY

A translation, beginning with 1956 issues of the physics sections of *Doklady Akademii Nauk SSSR*, the proceedings of the USSR Academy of Sciences. All-science journal offering four-page reports of recent research in physics and borderline subjects.

Vol. 6 comprising twelve issues, approx. 1500 pp.
\$35 domestic, \$38 foreign
Libraries* \$25 domestic, \$28 foreign
Single copies Vols. 1 and 2, \$4;
Vols. 3 through 5, \$8.00
Vol. 6 and subsequent issues, \$4.00

Soviet Physics—CRYSTALLOGRAPHY

A translation, beginning with 1957 issues of the journal *Kristallografiya* of the USSR Academy of Sciences. Experimental and theoretical papers on crystal structure, lattice theory, diffraction studies, and other topics of interest to crystallographers, mineralogists, and metallurgists.

Vol. 6 comprising six issues, approx. 1000 pp.
\$25 domestic, \$27 foreign
Libraries* \$15 domestic, \$17 foreign. Single copies, \$5

SOVIET ASTRONOMY—AJ

A translation, beginning with 1957 issues of *Astromicheskii Zhurnal* of the USSR Academy of Sciences. Covers various problems of interest to astronomers and astrophysicists including solar activity, stellar studies, spectroscopic investigations of radio astronomy.

Vol. 5 comprising six issues, approx. 1100 pp.
\$25 domestic, \$27 foreign
Libraries* \$15 domestic, \$17 foreign. Single copies, \$5

Soviet Physics—USPEKHI

A translation, beginning with September, 1958, issue of *Uspekhi Fizicheskikh Nauk* of the USSR Academy of Sciences. Offers reviews of recent developments comparable in scope and treatment to those carried in *Reviews of Modern Physics*. Also contains reports on scientific meetings within the Soviet Union, book reviews, and personalia.

Vol. 4 comprising six issues, approx. 1700 pp.
(Contents limited to material from Soviet sources)
\$45 domestic, \$48 foreign
Libraries* \$30 domestic, \$33 foreign. Single copies, \$8

*For libraries of nonprofit academic institutions.